Two-body decay kinematics

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Abstract

We calculated the 4-momenta of the daughter particles of the decay process $0 \to 1 + 2$ in the rest frame of the parent particle 0 and in the lab frame. The result has been applied to $\Lambda \to p\pi$ and $X \to \Lambda\bar{\Lambda}$ decays.

$0 \to 1+2$ decay in the rest frame of 0 In the rest frame of the particle 0, the 4-momentum of the particles are related by $P = p_1 + p_2$ where $P = (M, \vec{0})$. Hence, we can write $p_2 = P - p_1$, we obtain

$$
\begin{aligned}
p_2^2 = (P - p_1)^2 &= P^2 - 2P \cdot p_1 + p_1^2, \\
m_2^2 = M^2 - 2ME_1 + m_1^2 \Rightarrow E_1 &= \frac{M^2 + m_1^2 - m_2^2}{2M}, \\
E_2 = M - E_1 \Rightarrow E_2 &= \frac{M^2 + m_2^2 - m_1^2}{2M}.
\end{aligned}
$$

From $p_1 = \sqrt{E_1^2 - m_1^2}$ and $p_2 = \sqrt{E_2^2 - m_2^2}$, where $p_1 = |\vec{p_1}|$ and $p_2 = |\vec{p_2}|$, we have

$$
\begin{aligned}
p_1 &= \frac{\sqrt{(M^2 + m_1^2 - m_2^2)^2 - 4M^2 m_1^2}}{2M}, \\
p_2 &= \frac{\sqrt{(M^2 + m_2^2 - m_1^2)^2 - 4M^2 m_2^2}}{2M}.
\end{aligned}
$$

$\Lambda \to p\pi$ decay Given that $m_\Lambda = 1.115683$ GeV, $m_\pi = 0.1396$ GeV and $m_p = 0.938$ GeV, we have

$$\begin{aligned}
E_p &= \frac{m_\Lambda^2 + m_p^2 - m_\pi^2}{2m_\Lambda} = 0.943 \text{ GeV}, \\
E_\pi &= \frac{m_\Lambda^2 + m_\pi^2 - m_p^2}{2m_\Lambda} = 0.172 \text{ GeV}, \\
p_p &= \sqrt{E_p^2 - m_p^2} = 0.1 \text{ GeV}, \\
p_\pi &= \sqrt{E_\pi^2 - m_\pi^2} = 0.1 \text{ GeV}
\end{aligned}$$

in the rest frame of Λ.

$X \to \Lambda\bar{\Lambda}$ decay In the rest frame of X, we replace m_1 and m_2 by m_Λ. In addition, we know that $Q^2 = -(p_\Lambda - p_{\bar{\Lambda}})^2$ and by symmetry $E_\Lambda = E_{\bar{\Lambda}}$ and $\vec{p}_\Lambda = -\vec{p}_{\bar{\Lambda}}$. Hence we have $Q = |p_\Lambda| + |p_{\bar{\Lambda}}| = 2p_\Lambda$ and we obtain

$$\begin{aligned}
E_\Lambda = E_{\bar{\Lambda}} &= \frac{M}{2}, \\
p_\Lambda = p_{\bar{\Lambda}} &= \frac{1}{2}\sqrt{M^2 - 4m_\Lambda^2} = \frac{Q}{2}.
\end{aligned}$$

We have $Q^2 = -(p_\Lambda - p_{\bar{\Lambda}})^2 = M^2 - 4m_\Lambda^2$ where M is the invariant mass of X.

Boosting $0 \to 1+2$ decay to the lab frame Assuming particle 1 decays with an angle θ with 0 in the rest frame of 0.

$$\begin{aligned}
E_1^0 &= \frac{M^2 + m_1^2 - m_2^2}{2M}, \\
E_2^0 &= \frac{M^2 + m_2^2 - m_1^2}{2M}, \\
p_{1x}^0 &= -p_{2x}^0 = p_1^0 \sin\theta^0, \\
p_{1z}^0 &= -p_{2z}^0 = p_1^0 \cos\theta^0.
\end{aligned}$$

Now, we assume particle 0 has momentum P in the lab frame. We boost the particles to the moving frame of 0,

$$
\begin{aligned}
E &= \sqrt{P^2 + M^2}, \\
\gamma &= \frac{E}{M}, \\
v &= \frac{P}{E}.
\end{aligned}
$$

Using Lorentz transformation, we obtain

$$
\begin{aligned}
p_{1x} &= p^0_{1x} = p^0_1 \sin\theta^0, \\
p_{1z} &= \gamma(p^0_{1z} + vE^0_1), \\
\tan\theta_1 &= \frac{p_{1x}}{p_{1z}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
p_{2x} &= p^0_{2x} = -p^0_1 \sin\theta^0, \\
p_{2z} &= \gamma(p^0_{2z} + vE^0_2), \\
\tan\theta_2 &= \frac{p_{2x}}{p_{2z}}.
\end{aligned}
$$

$\Lambda \to p\pi$ decay We replace P by p_Λ, M by m_Λ, E_1 by E_p, E_2 by E_π, p_1 by p_p and p_2 by p_π.

$$
\begin{aligned}
E_\Lambda &= \sqrt{p^2_\Lambda + m^2_\Lambda}, \\
\gamma_\Lambda &= \frac{E_\Lambda}{m_\Lambda} = \frac{\sqrt{p^2_\Lambda + m^2_\Lambda}}{m_\Lambda}, \\
v_\Lambda &= \frac{p_\Lambda}{E_\Lambda} = \frac{p_\Lambda}{\sqrt{p^2_\Lambda + m^2_\Lambda}}.
\end{aligned}
$$

Hence, the decay angle θ_p in the lab frame is given by

$$\begin{aligned}
p_{px} &= 0.1 \sin \theta_p^0 \text{ GeV}, \\
p_{pz} &= \gamma_\Lambda (0.1 \cos \theta_p^0 + 0.943 v_\Lambda) \text{ GeV}, \\
\tan \theta_p &= \frac{p_{px}}{p_{pz}} = \frac{0.1 \sin \theta_p^0}{\gamma_\Lambda (0.1 \cos \theta_p^0 + 0.943 v_\Lambda)}, \\
p_p &= \sqrt{p_{px}^2 + p_{pz}^2}, \\
p_{\pi x} &= -0.1 \sin \theta_p^0 \text{ GeV}, \\
p_{\pi z} &= \gamma_\Lambda (-0.1 \cos \theta_p^0 + 0.172 v_\Lambda) \text{ GeV}, \\
\tan \theta_\pi &= \frac{p_{\pi x}}{p_{\pi z}} = \frac{-0.1 \sin \theta_p^0}{\gamma_\Lambda (-0.1 \cos \theta_p^0 + 0.172 v_\Lambda)}, \\
p_\pi &= \sqrt{p_{\pi x}^2 + p_{\pi z}^2}.
\end{aligned}$$

The functions have been plotted in **Figure 1** and **Figure 2**.

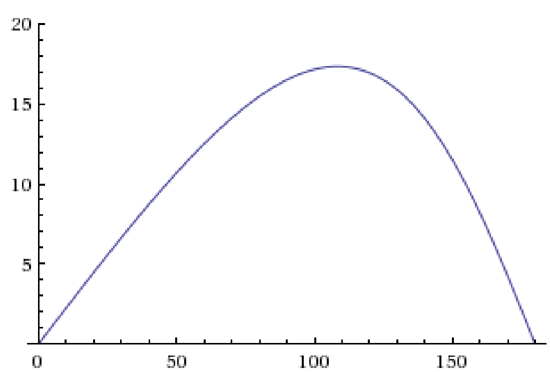

Figure 1: θ_p vs θ_p^0 distribution from 0 to 180 degrees for $p_\Lambda = 0.4$ GeV.

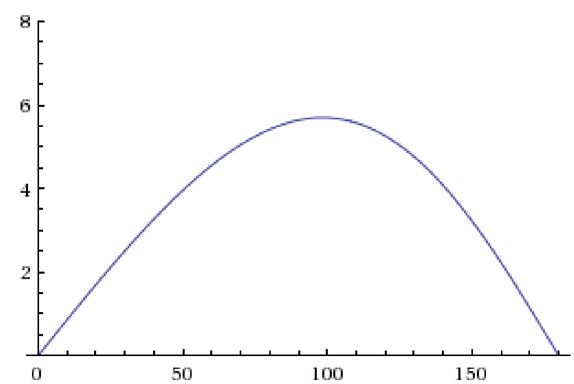

Figure 2: θ_p vs θ_p^0 distribution from 0 to 180 degrees for $p_\Lambda = 1.2$ GeV.

Example 1 For $p_\Lambda = 0.4$ GeV, we have $\gamma_\Lambda = 1.062$ and $v_\Lambda = 0.337$. We obtain

$$
\begin{aligned}
\tan\theta_p &= \frac{0.1\sin\theta_p^0}{1.062(0.1\cos\theta_p^0 + 0.943\times 0.337)} \\
&= \frac{0.942\sin\theta_p^0}{\cos\theta_p^0 + 3.178}, \\
p_p &= \sqrt{p_{px}^2 + p_{pz}^2} = \sqrt{(0.1\sin\theta_p^0)^2 + 1.128(0.1\cos\theta_p^0 + 0.318)^2}, \\
\tan\theta_\pi &= \frac{-0.1\sin\theta_p^0}{1.062(-0.1\cos\theta_p^0 + 0.172\times 0.337)}, \\
&= \frac{0.942\sin\theta_p^0}{-\cos\theta_p^0 + 0.58}, \\
p_\pi &= \sqrt{p_{\pi x}^2 + p_{\pi z}^2} = \sqrt{(-0.1\sin\theta_p^0)^2 + 1.128(-0.1\cos\theta_p^0 + 0.058)^2}.
\end{aligned}
$$

For $\theta_p^0 \sim 0$, we have $\cos\theta_p^0 \sim 1$ and $\sin\theta_p^0 \sim 0$,

$$
\begin{aligned}
p_p &\sim \sqrt{1.128}\times 0.418 = 0.614 \text{ GeV} \\
p_\pi &\sim \sqrt{1.128}\times 0.042 = 0.045 \text{ GeV}.
\end{aligned}
$$

For $\theta_p^0 \sim \frac{\pi}{2}$, we have $\cos\theta_p^0 \sim 0$ and $\sin\theta_p^0 \sim 1$,

$$
\begin{aligned}
p_p &\sim \sqrt{0.1^2 + 1.128\times 0.318^2} = 0.35 \text{ GeV} \\
p_\pi &\sim \sqrt{0.1^2 + 1.128\times 0.058^2} = 0.12 \text{ GeV}.
\end{aligned}
$$

For $\theta_p^0 \sim \pi$, we have $\cos\theta_p^0 \sim -1$ and $\sin\theta_p^0 \sim 0$,

$$
\begin{aligned}
p_p &\sim \sqrt{1.128}\times 0.218 = 0.232 \text{ GeV} \\
p_\pi &\sim \sqrt{1.128}\times 0.158 = 0.167 \text{ GeV}.
\end{aligned}
$$

Example 2 For $p_\Lambda = 1.2$ GeV, we have $\gamma_\Lambda = 1.469$ and $v_\Lambda = 0.732$. We obtain

$$
\begin{aligned}
\tan\theta_p &= \frac{0.1\sin\theta_p^0}{1.469(0.1\cos\theta_p^0 + 0.943\times 0.732)} \\
&= \frac{0.681\sin\theta_p^0}{\cos\theta_p^0 + 6.903}, \\
p_p &= \sqrt{p_{px}^2 + p_{pz}^2} = \sqrt{(0.1\sin\theta_p^0)^2 + 2.158(0.1\cos\theta_p^0 + 0.69)^2}, \\
\tan\theta_\pi &= \frac{-0.1\sin\theta_p^0}{1.469(-0.1\cos\theta_p^0 + 0.172\times 0.732)} \\
&= \frac{0.681\sin\theta_p^0}{-\cos\theta_p^0 + 0.857}, \\
p_\pi &= \sqrt{p_{\pi x}^2 + p_{\pi z}^2} = \sqrt{(-0.1\sin\theta_p^0)^2 + 2.158(-0.1\cos\theta_p^0 + 0.126)^2}.
\end{aligned}
$$

For $\theta_p^0 \sim 0$, we have $\cos\theta_p^0 \sim 1$ and $\sin\theta_p^0 \sim 0$,

$$
\begin{aligned}
p_p &\sim \sqrt{2.158}\times 0.79 = 1.16 \text{ GeV} \\
p_\pi &\sim \sqrt{2.158}\times 0.026 = 0.038 \text{ GeV}.
\end{aligned}
$$

For $\theta_p^0 \sim \frac{\pi}{2}$, we have $\cos\theta_p^0 \sim 0$ and $\sin\theta_p^0 \sim 1$,

$$
\begin{aligned}
p_p &\sim \sqrt{0.1^2 + 2.158\times 0.69^2} = 1.02 \text{ GeV} \\
p_\pi &\sim \sqrt{0.1^2 + 2.158\times 0.126^2} = 0.21 \text{ GeV}.
\end{aligned}
$$

For $\theta_p^0 \sim \pi$, we have $\cos\theta_p^0 \sim -1$ and $\sin\theta_p^0 \sim 0$,

$$
\begin{aligned}
p_p &\sim \sqrt{2.158}\times 0.59 = 0.87 \text{ GeV} \\
p_\pi &\sim \sqrt{2.158}\times 0.226 = 0.33 \text{ GeV}.
\end{aligned}
$$

Summary From **Example 1** and **Example 2**, it has been shown that the kinematics of the daughter particles in the lab frame depends heavily on the kinematics of the parent particle as well as the decay angle. In the case of the decay process $\Lambda \rightarrow p\pi$, π always has lower momentum than p in the lab frame. With a track reconstruction p_T threshold, say 0.15 GeV, events are removed unevenly in the decay angle phase space, with more events removed when θ_p^0 is close to 0 due to the lower momentum of π. This may explain why the correlation introduced by the detector effect alone, without any contribution from the spin correlation effect, produced the observed structure when plotted as a function of Q and why the reconstructed decay angle distribution is skewed to the negative side.

P.S. For simplicity, we did not first boost Λ and $\bar{\Lambda}$ to the center-of-mass frame for the helicity basis.