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Aug. 26, 2015

Relativistic Kinematics II

Table of Contents

1 Using 4-Vectors

Relativistic kinematics problems are greatly simplified by using 4-vectors, which provide useful notational convenience and powerful methods for evaluation, including the freedom to select a reference frame to simplify evaluation. Additionally, for any 4-momentum p_A,

$$p_A^2 \equiv E_A^2 - \mathbf{p}_A^2 = m_A^2 .$$

A 4-momentum equation automatically takes into account conservation of energy and momentum, i.e. 4 constraints. For example, if a particle P decays into three daughters, we write the 4-momentum equation $P = p_1 + p_2 + p_3$, which is shorthand for

$$\begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} + \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} + \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix}$$

Similarly, for a scattering process of two particles into two particles, $1+2 \to 3+4$, we write the conservation equation $p_1 + p_2 = p_3 + p_4$, which stands for the 4 equations

$$\begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} + \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} = \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix} + \begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}$$

2 Decay of a Particle to Two Daughters

Consider the decay of a particle with mass M to two particles of mass m_1 and m_2 in the rest frame of the parent particle. The two final particles are defined by 8 components (2 energies and 6 momenta). However, for daughter particles of well-defined masses, we have two constraints: $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$, leaving us with 6 variables. Application of 4-momentum conservation leaves us with just 2 degrees of freedom. Since the two daughter particles must be emitted back to back in the parent rest frame (3-momentum conservation), we can choose the parameters to be two angles (e.g., θ and ϕ) defining the direction of one of the daughters.

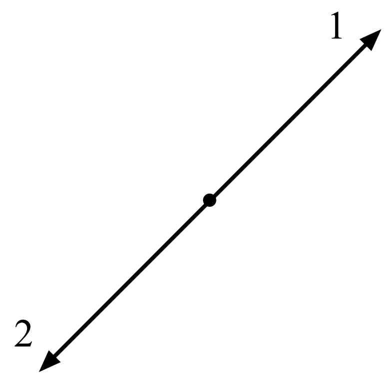

Figure 1: Two body decay in rest frame of parent particle

We write the 4-momentum equation $P = p_1 + p_2$, where $P = (M, \mathbf{0})$ in the parent's rest frame. Let's find the energy and momentum for particle 1. Since we don't need particle 2's information yet, we write $p_2 = P - p_1$ and square p_2 to get only its mass information:

$$\begin{aligned} p_2^2 &= (P - p_1)^2 = P^2 - 2P \cdot p_1 + p_1^2 \\ m_2^2 &= M^2 - 2ME_1 + m_1^2 \end{aligned}$$

Here I used $P \cdot p_1 = ME_1$, taking advantage of the fact that we are evaluating in the parent rest frame. The only unknown is E_1, which we solve for:

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}$$

An identical evaluation can be done for particle 2 by switching indices or by using $E_2 = M - E_1$

$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}$$

We can also work out the momentum from $p_1 = \sqrt{E_1^2 - m_1^2}$, yielding

$$p_1 = \frac{\sqrt{\left(M^2 + m_1^2 - m_2^2\right)^2 - 4M^2 m_1^2}}{2M} = \frac{\sqrt{\left(M^2 - m_1^2 - m_2^2\right)^2 - 4m_1^2 m_2^2}}{2M}$$

The far RHS expression is clearly symmetric under $1 \leftrightarrow 2$ interchange so $p_1 = p_2$, as required by 3-momentum conservation. Note that I use *p* sometimes to represent $p = |\mathbf{p}|$ and sometimes to represent the 4-momentum. The meaning should be clear from the context.

2.1 Example 1: $\pi^+ \to \mu^+ + \nu_\mu$

This decay mode accounts for almost muon decays. Here $m_\pi = 0.1396$ GeV, $m_\mu = 0.1057$ GeV and $m_\nu = 0$. We obtain

$$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = 0.1098\,\text{GeV}$$

$$E_\nu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = 0.0298\,\text{GeV}$$

$$p_\mu = p_\nu = E_\nu = 0.0298\,\text{GeV}$$

2.2 Example 2: $K^{*-} \to K^- + \pi^0$

Here $m_{K^*} = 0.8917$ GeV, $m_{K^-} = 0.4937$ GeV and $m_{\pi^0} = 0.1350$ GeV. We obtain

$$E_{K^-} = \frac{m_{K^*}^2 + m_{K^-}^2 - m_{\pi^0}^2}{2m_{K^*}} = 0.5723\,\text{GeV}$$

$$E_{\pi^0} = E_{K^*} - E_{K^-} = 0.3194\,\text{GeV}$$

$$p_{K^-} = p_{\pi^0} = \sqrt{E_{K^-}^2 - m_{K^-}^2} = 0.2895\,\text{GeV}$$

Note the use of conservation laws in determining the π^0 energy and momenta.

2.3 Example 3: Impossibility of $e^- \to e^- + \gamma$

We can ask under what circumstances a high-energy electron can decay into an electron plus a photon. The 4-momentum conservation equation is $p_e = p_{e'} + p_\gamma$. Since we don't know anything about the final state electron we isolate its 4-momentum and square it to eliminate its kinematic quantities (a trick we resort to time and again) to obtain:

$$p_{e'}^2 = m_e^2 = \left(p_e - p_\gamma\right)^2 = m_e^2 - 2p_e \cdot p_\gamma + 0$$

which yields $p_e \cdot p_\gamma = E_\gamma \left(E_e - p_e \cos\theta\right) = 0$ using $E_\gamma = p_\gamma$ for the massless photon. This equation cannot be satisfied unless $E_\gamma = 0$.

A simpler way to see this result is to evaluate it in the initial electron rest frame, taking advantage of the fact that a physical process must take place in all coordinate systems. Conservation of energy in this frame ($m_e = E_{e'} + E_\gamma$) clearly shows the impossibility of the reaction unless $E_\gamma = 0$.

3 Boosting Daughter Particles From Parent Rest Frame

The energies and momenta calculated for two-body decay in Section 2 were determined in the parent particle CM frame. However, we frequently need to find their values in the frame where the parent particle is moving, e.g. the detector frame.

Let's use the $K^{*-} \to K^- + \pi^0$ example from Section 2.2 when the K^{*-} is moving with a momentum of 5.5 GeV. We assume that the K^- is emitted at $\theta^* = 55°$ relative to the K^{*-} direction of motion (measured in the K^{*-} rest frame).

Let the K^{*-} be moving along the +z direction and assume the decay happens in the x–z plane with p_K being the K^- momenta. In the K^{*-} rest frame, using the results of Section 2.2,

Pk=0.2895 VeV

$$\begin{aligned} {}_x &= p_K^* \sin\theta^* = 0.2371 & \qquad p_{\pi x}^* &= -p_{K x}^* = -0.2371 \\ p_{K z}^* &= p_K^* \cos\theta^* = 0.1661 & \qquad p_{\pi z}^* &= -p_{K z}^* = -0.1661 \\ E_K^* &= 0.5723 & \qquad E_\pi^* &= m_{K^{*-}} - E_K^* = 0.3194 \end{aligned}$$

Now we boost to the frame where the parent K^{*-} is moving, using

$$E_{K^*} = \sqrt{p_{K^*}^2 + m_{K^*}^2} = \sqrt{5.5^2 + 0.8917^2} = 5.5718$$

$$\gamma_{K^*} = \frac{E_{K^*}}{m_{K^*}} = 6.2485$$

$$v_{K^*} = \frac{p_{K^*}}{E_{K^*}} = 0.9871$$

Then using the LT, we obtain

$$p_{Kx} = p^*_{Kx} = 0.2371\,\text{GeV}$$

$$p_{Kz} = \gamma_{K^*}\left(p^*_{Kz} + v_{K^*}E^*_K\right) = 6.2485\left(0.1661 + 0.9871 \times 0.5723\right) = 4.567\,\text{GeV}$$

$$\tan\theta_K = \frac{p_{Kx}}{p_{Kz}} = \frac{0.2371}{4.567} = 0.0519$$

$$\theta_K \simeq 2.97°$$

We can do a similar calculation for the π^0, for which $p^*_{\pi x} = -p^*_{Kx}$ and $p^*_{\pi z} = -p^*_{Kz}$.

$$p_{\pi x} = p^*_{\pi x} = -0.2371\,\text{GeV}$$

$$p_{\pi z} = \gamma_{K^*}\left(p^*_{\pi z} + v_{K^*}E^*_\pi\right) = 6.2485\left(-0.1661 + 0.9871 \times 0.3194\right) = 0.9325\,\text{GeV}$$

$$\tan\theta_\pi = \frac{p_{\pi x}}{p_{\pi z}} = \frac{-0.2371}{0.9325} = -0.2543$$

$$\theta_\pi \simeq -14.3°$$

Thus the π^0 is boosted forward, 14 degrees relative to the K^{*-} direction, while the K^- is boosted forward, 3 degrees relative to the K^{*-} direction but on the opposite side (see Figure 2). As a check, $p_{Kz} + p_{\pi z} = p_{K^*} = 5.5$ GeV, as required by momentum conservation.

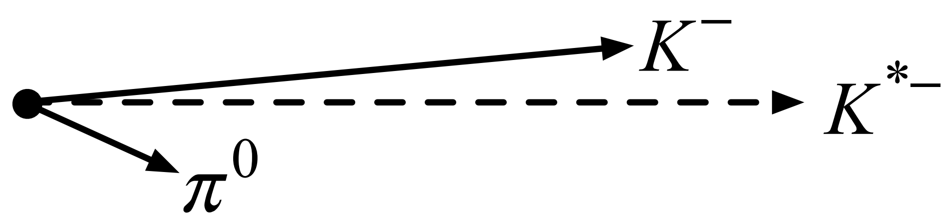

Figure 2: $K^{*-} \to K^- + \pi^0$ decay in the frame where the K^{*-} is moving with momentum 5.5 GeV/c. The decay products are Lorentz boosted in the K^{*-} direction. A parent with high momentum will have decay products having large momentum in the parent direction with transverse components unchanged, leading to small angles relative to the parent direction.

For completeness, we plot in Figure 3 the angle of the two particles in the lab frame for all decay angles in the K^{*-} center of mass for a K^{*-} having momentum 5.5 GeV/c. Note the following:

- The angle on the *x*-axis is the decay angle of the K^-. Angles in the forward direction correspond to forward emission of the K^- and backward emission of the π^0.
- The angles relative to the K^{*-} direction are small. If the K^{*-} had twice the energy the angles would be roughly half as large. That's because the transverse momenta are the same, but the longitudinal momenta are boosted by a gamma factor twice as large.

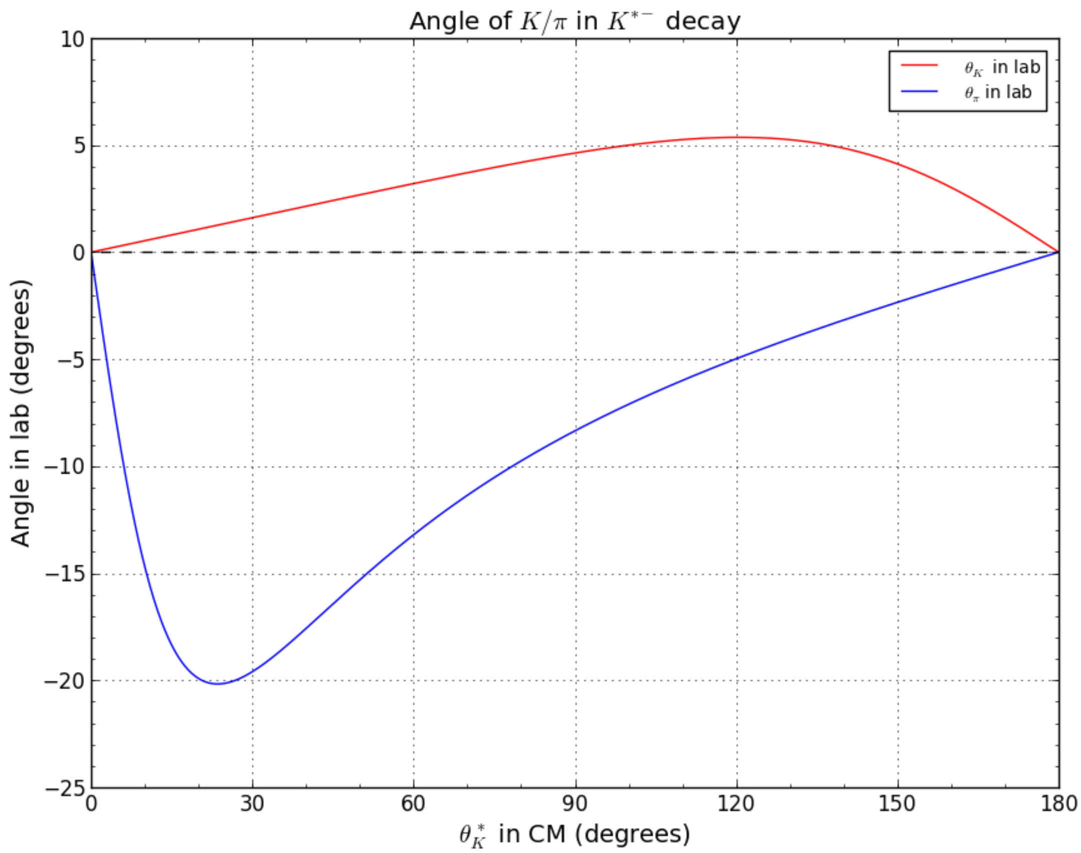

Figure 3: Direction of daughters from $K^{*-} \to K^- + \pi^0$ decay in the frame where the K^{*-} is moving with momentum 5.5 GeV/c. The x value is the angle of emission (in degrees) of the K^- in the K^{*-} rest frame, while the y value is the angle (in degrees) of each daughter relative to the K^{*-} direction.

4 Scattering Processes

In a scattering processes, two particles A and B interact and either fly away at different angles (elastic scattering) or produce new particles (inelastic scattering). We write the general process as A + B → 1 + 2 + 3 + ... N, where conservation of 4-momentum implies

$$p_A + p_B = p_1 + p_2 + \cdots + p_N .$$

4.1 Fixed target and colliding beam experiments: Lab and CM frames

Experimental setups for carrying out scattering experiments are of two types: (1) **fixed target**, where a beam of particles strikes particles at rest and (2) **colliding beams**, where counterrotating beam particles strike one another head on. Fixed target experiments are the easiest to set up and have been used for decades. They have the advantage of high collision rates ("luminosity") and the ability to utilize a wide variety of beam and target particles. Colliding beam accelerators have the advantage of being able to reach extremely high effective energies (because particles collide head on) but have much lower luminosities.

Two standard Lorentz frames, corresponding to each of these setups, are in wide use. The LAB frame is the coordinate system in which the beam particle is moving and the target is at rest. The

Center of Momentum (CM) frame is the coordinate system in which the net momentum is zero, i.e., in which the beam and target particle move with equal and opposite momenta. Most calculations of elementary processes are performed in the CM frame, although it is straightforward to use the LT to transform the results to other frames.

In describing the energy available in a collision, it has become standard to use the Lorentz invariant quantity $s = \left(p_A + p_B\right)^2$. We immediately see from evaluating it in the CM frame that $s = \left(E_A^* + E_B^*\right)^2 = E_{CM}^2$ or the center of mass energy squared. $\sqrt{s}$ is thus the largest total mass that can produced in a collision process. You will frequently see s and $\sqrt{s}$ used in papers and even titles of papers, e.g. "Measurement of W production in CMS at $\sqrt{s} = 7.0$ TeV", which refers to measurements measured in the CMS detector using colliding proton beams of E = 3.5 TeV.

In the CM frame, $s = 4E^{*2}$ for identical mass beam particles each with energy E^*. In the LAB frame $s = \left(p_A + p_B\right)^2 = m_A^2 + 2E_A m_B + m_B^2$, using $p_A \cdot p_B = E_A m_B$ because B is at rest. For a given beam energy, *all* the energy in the colliding beams is available to produce new particles. In contrast, most of the energy in a fixed target reaction is wasted because conservation of momentum requires the final state products to carry away the initial beam momentum and thus kinetic energy. As beam energies become very large, $E \gg m_A, m_B$, the advantage of using colliding beams to reach high center of mass energies becomes quite dramatic:

$$\sqrt{s} \to \sqrt{2Em_B} \qquad \text{Fixed target}$$

$$\sqrt{s} \to 2E \qquad \text{Colliding beams}$$

The table below shows $\sqrt{s}$ values for proton-proton fixed target collisions at various lab energies. Note the incredibly large fixed target energies that would be required to reach the energies available in current colliding beam experiments.

Lab E (GeV) (Fixed target p-p)	$\sqrt{s} = \sqrt{2m_p^2 + 2Em_p}$ (GeV)
5	3.34
10	4.53
50	9.78
100	13.8
500	30.7
1000	43.3
5000	96.9
10000	137
2.0×10^6	1960 (Tevatron)
3.4×10^7	8000 (LHC, 2012)

1.0×10^{8}	14000 (LHC design)
$\sim2\times10^{11}$ (highest E cosmic)	~300,000

4.2 Threshold energy for reactions: Discovery of the antiproton

For particle production to take place there must be sufficient energy in the incoming particles A and B. The total center of mass energy must clearly be at least as large as the sum of the rest masses of all final state particles, or $\sqrt{s}\geq\sum m_i$. This condition imposes simple requirements on the beam energy for both fixed target and colliding beam experiments.

Consider the classic 1955 experiment to find the antiproton at the Bevatron at Berkeley, where Emilio Segrè, Owen Chamberlain and their colleagues looked for the reaction $p+p\to p+p+\overline{p}+p$. A minimum of 4 particles are expected in the final state because there are two baryons in the initial state, which by baryon conservation requires 3 baryons to accompany the antiproton. Thus we have $s_{\min}=\left(4m_p\right)^2=16m_p^2=2m_p^2+2m_pE_{\min}$ for the fixed target configuration, yielding $E_{\min}=7m_p\simeq 6.568$ GeV and $p_{\min}=\sqrt{E_{\min}^2-m_p^2}=\sqrt{48}m_p=6.501$ GeV.

One can easily find the velocity of the final state particles at this minimum energy. The final state particles are at rest in the CM frame (why?) and thus they all move in the LAB frame at the same velocity as the CM frame. Let the momentum and total energy of the CM system measured in the lab frame be called p_C and E_C. Then using $p_C=p_{\min}=\sqrt{48}m_p$ and $E_C=E_{\min}+m_p=8m_p$, we obtain the velocity of the CM frame measured in the lab, $v_C=\sqrt{48}/8=0.866$. Segrè and his colleagues used Cerenkov detectors to look for particles moving with this velocity to reduce the enormous background from other kinds of collisions.

5 Advanced topics

5.1 Kinematic Limits in 3-Body Decay

For a particle decaying to three daughters, the three final particles can have a range of momenta and energies that satisfy the 4-momentum conservation laws (5 independent parameters describe the decay). However, 4-momentum conservation imposes limits on the ranges of these values so that we can find the range of possible energies for, say, particle 1. The minimum energy is clearly m_1 (particle 1 at rest) where the particle momenta are shown in Figure 4.

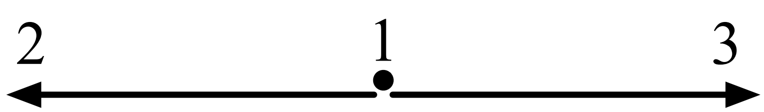

Figure 4: Configuration of momenta for three body decay where particle 1 at rest.

To calculate particle 1's maximum energy, think of particles 2 and 3 forming a single system Q, so that the decay can be thought of as a decay into particle 1 + "particle Q". Clearly the minimum mass of Q is m_1+m_2, where particles 2 and 3 are at rest in the Q rest frame and thus move together at the same velocity in the rest frame of the parent. The configuration is shown in Figure 5.

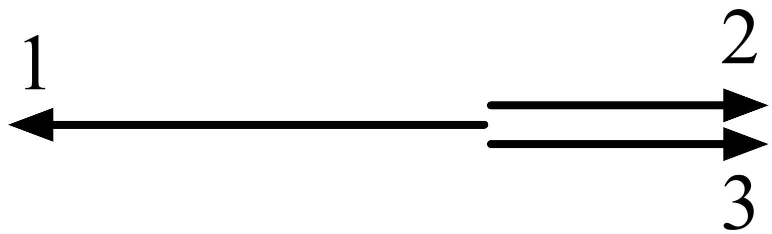

Figure 5: Configuration of momenta for three body decay where particle 1 has maximum energy.

Using the 2-body energy formula, the minimum and maximum values of E_1 are therefore

$$m_1 \le E_1 \le \frac{M^2 + m_1^2 - \left(m_2 + m_3\right)^2}{2M}$$

Similar expressions can be written for particles 2 and 3.

5.2 Ultra-high energy cosmic rays: The GZK cutoff

Cosmic rays are high-energy particles having extrastellar origins. They are primarily protons with some admixture of helium and other nuclei. The kinetic energy spectrum (measured up to ~300 TeV in a number of experiments) is shown in Figure 6[1] for nuclear species ranging from protons to iron. Although the abundance of each species falls dramatically with atomic mass, the kinetic energy distributions have similar shapes, falling like $dN/dK \sim K^{-2.7}$ at very high energy.

The proton spectrum has been measured up to energies of ~10^{20} eV (10^8 TeV) using ground-based air shower arrays to capture energetic but rare events above 10^{15} eV. Note that the CM energy for a cosmic ray particle at $E = 10^{20}$ eV striking a proton is approximately $\sqrt{s} \simeq \sqrt{2Em_p} \approx 430$ TeV, far higher than the LHC! It's believed that the origins of ultra-high energy cosmic rays ($E > 10^{18}$ eV) are extragalactic, perhaps correlated with AGNs (active galactic nuclei).

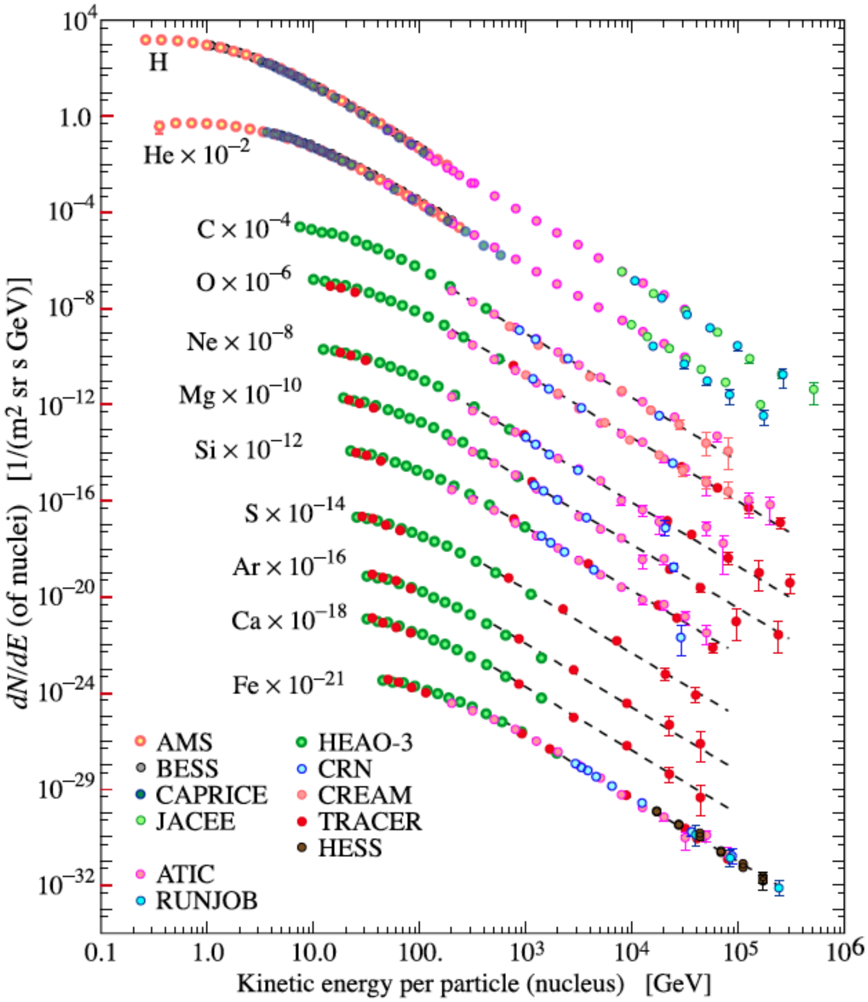

Figure 6: Primary cosmic ray spectra for different nuclei vs K (kinetic energy/nucleus) in GeV. Taken from PDG. The spectra fall approximately as $K^{-2.7}$ at high energies

In 1966, Greisen[2], Zatsepin and Kuzmin[3] (GZK) predicted that the high-energy cosmic ray spectrum would fall off sharply above a certain energy (calculated later to be $E \sim 6 \times 10^{19}$ eV or 60 EeV) because at that energy protons would begin interacting with photons from the cosmic microwave background (CMB) to produce Δ baryons through the reaction $\gamma + p \to \Delta^+ \to p + \pi^0$. The outgoing proton would have less energy than the initial proton because a fraction of the original energy would be taken by the outgoing pion. Thus ultra-energetic protons above the GZK cutoff energy would steadily shed energy through this reaction before they could travel significant intergalactic distances (less than 50 megaparsecs or 160 MLY). Of course, if these protons are produced inside the galaxy (typical size ~0.2 MLY) then reactions with CMB photons would not occur frequently enough to cause a significant reduction of the energy spectrum.

Let's calculate the proton energy threshold for the reaction $\gamma + p \to \Delta^+ \to p + \pi^0$. The CMB photon energies are distributed like a blackbody spectrum with temperature $T = 2.725$K (Figure 7). The distribution of photons vs energy is given by

$$\frac{dn_\gamma}{dE_\gamma} = \frac{1}{\pi^2 (\hbar c)^3} \frac{E_\gamma^2}{e^{E_\gamma / k_B T} - 1}$$

Integrating yields the total CMB photon number density of $n_\gamma = 410\,\text{cm}^{-3}$, an extremely low density. The peak of the distribution occurs at $1.590 k_B T = 0.375\,\text{meV}$ (1 meV = 0.001 eV) while the mean of the distribution is $2.70 k_B T = 0.635\,\text{meV}$.

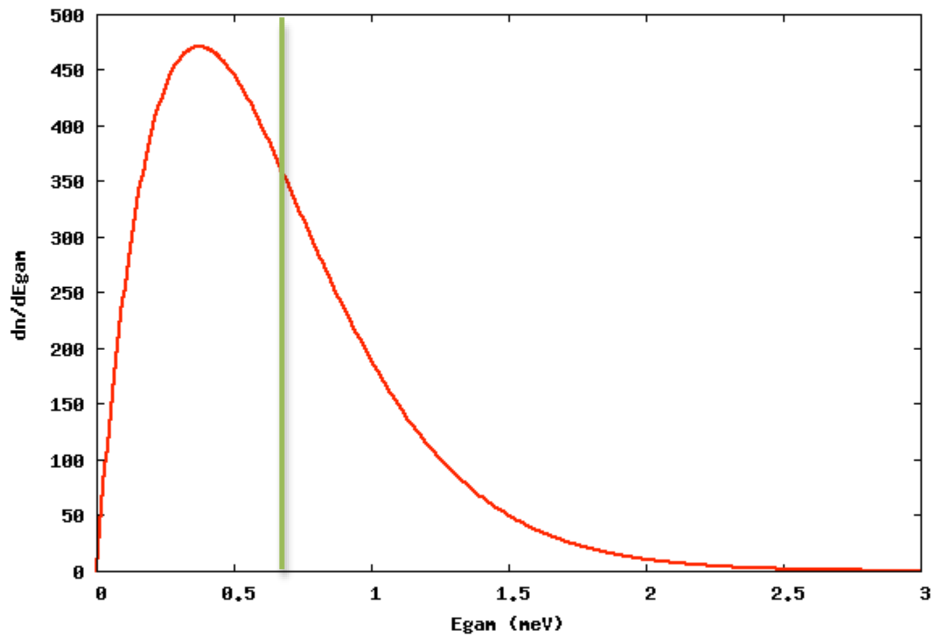

Figure 7: CMB photon distribution vs E_γ in meV. The peak of the distribution is at 0.375 meV and the average photon energy is 0.635 meV (vertical line).

It's easy to calculate the threshold energy for Δ production using the *s* invariant. For a highly relativistic proton hitting a photon, $s = (p_p + p_\gamma)^2 = m_p^2 + 2 p_p \cdot p_\gamma \simeq m_p^2 + 2 E_p E_\gamma (1 - \cos\theta)$, where θ is the angle between the momentum vectors of the proton and photon. This angle is random, but for now we assume a head-on collision, cos θ = –1. We will also assume the photon has the average energy of the CMB distribution or 0.635 meV = 6.35×10^{-4} eV. Finally, we need to put in the mass of the Δ. Although Figure 8 shows that the lowest mass Δ has a substantial width that extends to significantly lower values, we assume for now the center of the distribution or $s = m_\Delta^2$. The proton threshold energy to produce a Δ at 1.232 GeV is then

$$E_p = \frac{s - m_p^2}{4 E_\gamma} = \frac{1.232^2 - 0.938^2}{4 \times (0.635 \times 10^{-12})} = 2.5 \times 10^{11}\ \text{GeV} = 2.5 \times 10^{8}\ \text{TeV}$$

or $E_p = 2.5 \times 10^{20}\ \text{eV}$.

In a more advanced treatment, we would take into account non head-on collisions as well as the full CMB photon energy distribution (there are photons with energies substantially higher than the mean energy) and distribution of cross sections (which allows the reaction to take place at a center of mass energy lower than the Δ mass). In that treatment the reaction becomes significant even at the smaller proton energy value of $E_p \simeq 6 \times 10^{19}$ eV . Protons with energies above this threshold will suffer interactions with CMB photons and steadily lose energy over intergalactic distances. Thus if cosmic rays have intergalactic origins we expect to see a rapid fall-off of energies about this “GZK cutoff”.

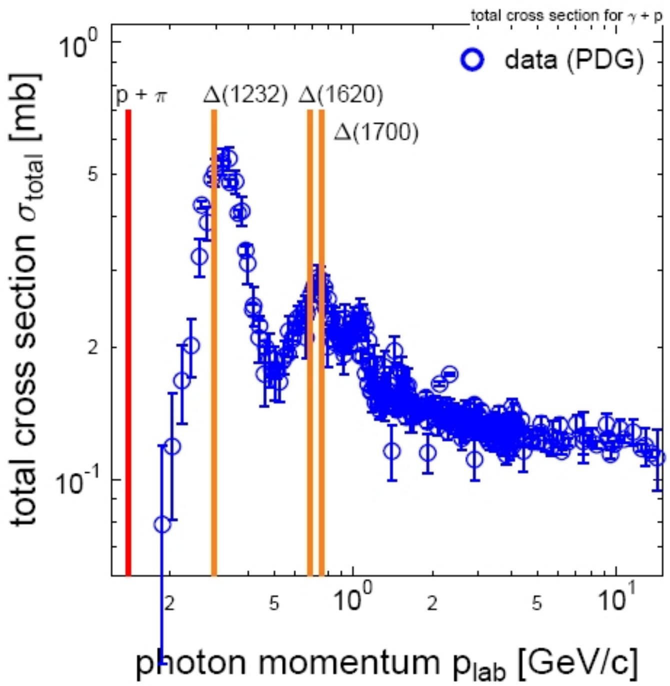

Figure 8: Total Cross section for the reaction γ + p vs E_γ, showing resonant peaks

What does the data show? Of the experiments which have measured the spectrum (Figure 9 and Figure 10), only AGASA does not show the cutoff. This was a subject of some controversy, although most physicists do not believe the older AGASA result.

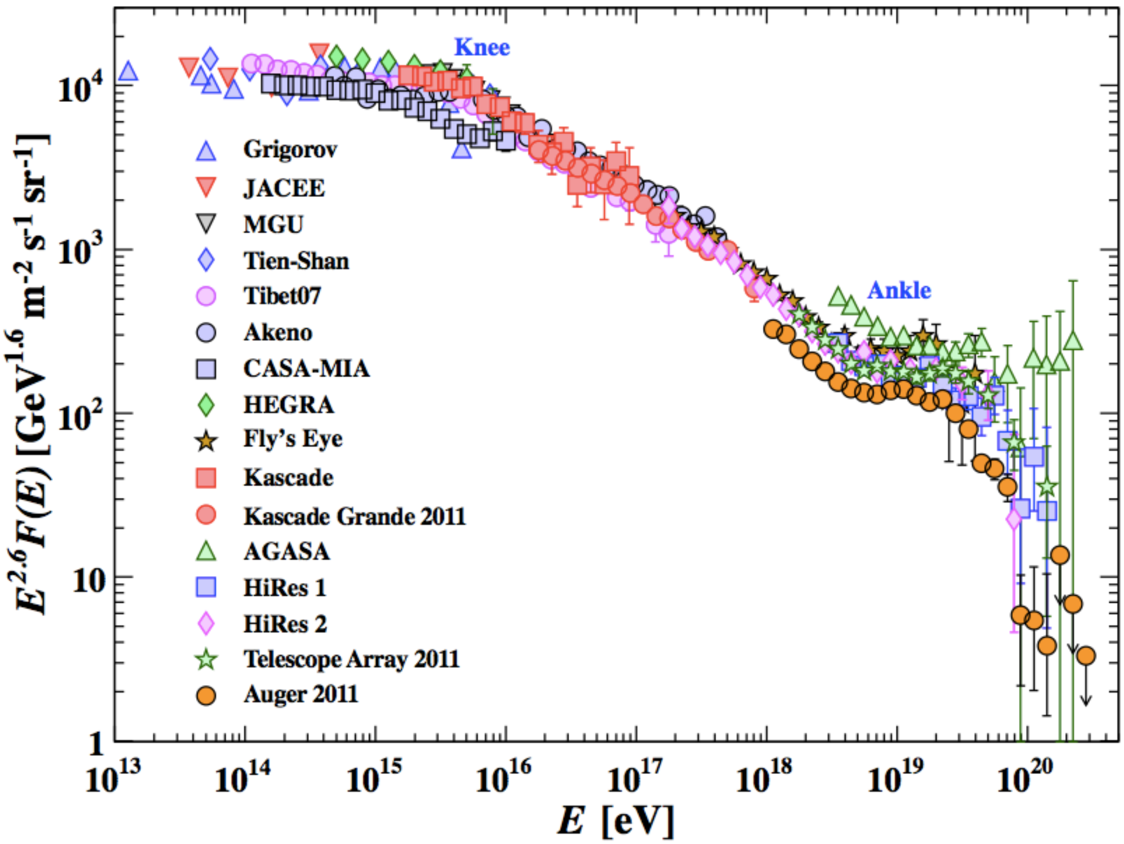

Figure 9: Measured ultra-high energy cosmic ray spectrum vs energy, showing data from several experiments.[1] Note that the spectrum has been multiplied by $E^{2.6}$ to present the data more clearly.

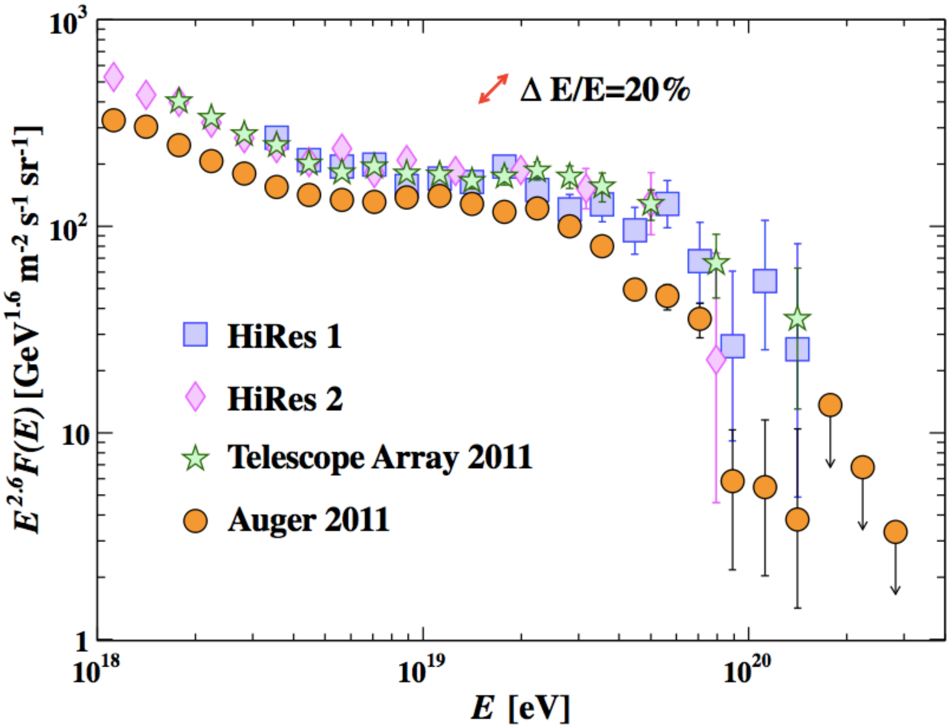

Figure 10: Ultra-high energy tail from Figure 9 showing data from the most recent cosmic energy measurements. These experiments show the cutoff at the GZK energy $E_p \simeq 6\times10^{19}$ eV.

5.3 Boosts in arbitrary directions (reference only)

It's essential when analyzing scattering and decay reactions to be able to boost 4-momenta to and from reference frames that are moving in arbitrary directions. This can be accomplished by rewriting the formulas for a boost along the z axis in a rotationally invariant way. The LT formulas transforming the 4-momentum components p' in a frame moving at velocity $+v$ relative to another frame along the z axis are

$$\begin{aligned}
E &= \gamma_v\left(E' + vp'_z\right) \\
p_x &= p'_x \\
p_y &= p'_y \\
p_z &= \gamma_v\left(p'_z + vE'\right)
\end{aligned}$$

where $\gamma_v = 1/\sqrt{1-v^2}$. Since p_z is merely the component along the velocity of the reference frame, we rewrite the LT using momentum components parallel to and perpendicular to the velocity:

$$\begin{aligned}
E &= \gamma_v\left(E' + vp'_{\parallel}\right) \\
p_{\perp} &= p'_{\perp} \\
p_{\parallel} &= \gamma_v\left(p'_{\parallel} + vE'\right)
\end{aligned}
\quad\Longrightarrow\quad
\begin{aligned}
E &= \gamma_v\left(E' + \mathbf{v}\cdot\mathbf{p}'\right) \\
\mathbf{p}_{\perp} &= \mathbf{p}'_{\perp} \\
\mathbf{p}_{\parallel} &= \gamma_v\left(\mathbf{p}'_{\parallel} + \mathbf{v}E'\right)
\end{aligned}$$

The momentum components $\mathbf{p}_{||}$ and $\mathbf{p}_{\perp}$ satisfy $\mathbf{p} = \mathbf{p}_{||} + \mathbf{p}_{\perp}$. We calculate them as

$$\mathbf{p}_{||} = \hat{\mathbf{v}}\left(\mathbf{p} \cdot \hat{\mathbf{v}}\right) = \mathbf{v}\left(\mathbf{p} \cdot \mathbf{v}\right) / v^2$$
$$\mathbf{p}_{\perp} = \mathbf{p} - \mathbf{p}_{||} = \mathbf{p} - \mathbf{v}\left(\mathbf{p} \cdot \mathbf{v}\right) / v^2$$

Using $\mathbf{p} = \mathbf{p}_{||} + \mathbf{p}_{\perp}$ and the above values of $\mathbf{p}_{||}$ and $\mathbf{p}_{\perp}$, we obtain the LT for a boost in an arbitrary direction

$$E = \gamma_v \left(E' + \mathbf{v} \cdot \mathbf{p}' \right)$$
$$\mathbf{p} = \mathbf{p}' - \frac{\mathbf{v}\left(\mathbf{p}' \cdot \mathbf{v}\right)}{v^2} + \gamma_v \left(\frac{\mathbf{v}\left(\mathbf{p}' \cdot \mathbf{v}\right)}{v^2} + \mathbf{v}E' \right)$$

Collecting terms and simplifying, using $v^2 = 1 - 1/\gamma_v^2$, yields the following useful formulas relating the new 4-momentum components p to the components p' in a frame moving with velocity $\mathbf{v}$.

$$E = \gamma_v \left(E' + \mathbf{v} \cdot \mathbf{p}' \right)$$
$$\mathbf{p} = \mathbf{p}' + \frac{\mathbf{v}\left(\mathbf{p}' \cdot \mathbf{v}\right)\gamma_v^2}{\left(\gamma_v + 1\right)} + \mathbf{v}\gamma_v E'$$

We can write the LT as a matrix equation that can be easily be coded in a computer program

$$p = \Lambda p'$$

$$\Lambda = \begin{pmatrix} \gamma_v & \gamma_v v_x & \gamma_v v_y & \gamma_v v_z \\ \gamma_v v_x & 1 + \dfrac{\gamma_v^2 v_x^2}{\gamma_v + 1} & \dfrac{\gamma_v^2 v_x v_y}{\gamma_v + 1} & \dfrac{\gamma_v^2 v_x v_z}{\gamma_v + 1} \\ \gamma_v v_y & \dfrac{\gamma_v^2 v_y v_x}{\gamma_v + 1} & 1 + \dfrac{\gamma_v^2 v_y^2}{\gamma_v + 1} & \dfrac{\gamma_v^2 v_y v_z}{\gamma_v + 1} \\ \gamma_v v_z & \dfrac{\gamma_v^2 v_z v_x}{\gamma_v + 1} & \dfrac{\gamma_v^2 v_z v_y}{\gamma_v + 1} & 1 + \dfrac{\gamma_v^2 v_z^2}{\gamma_v + 1} \end{pmatrix} = \begin{pmatrix} \gamma_v & \tilde{v}_x & \tilde{v}_y & \tilde{v}_z \\ \tilde{v}_x & 1 + \dfrac{\tilde{v}_x^2}{\gamma_v + 1} & \dfrac{\tilde{v}_x \tilde{v}_y}{\gamma_v + 1} & \dfrac{\tilde{v}_x \tilde{v}_z}{\gamma_v + 1} \\ \tilde{v}_y & \dfrac{\tilde{v}_y \tilde{v}_x}{\gamma_v + 1} & 1 + \dfrac{\tilde{v}_y^2}{\gamma_v + 1} & \dfrac{\tilde{v}_y \tilde{v}_z}{\gamma_v + 1} \\ \tilde{v}_z & \dfrac{\tilde{v}_z \tilde{v}_x}{\gamma_v + 1} & \dfrac{\tilde{v}_z \tilde{v}_y}{\gamma_v + 1} & 1 + \dfrac{\tilde{v}_z^2}{\gamma_v + 1} \end{pmatrix}$$

where $\tilde{v}_i = \gamma_v v_i$.

References

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[1] Measurements plotted from references in http://pdg.lbl.gov/2013/reviews/rpp2012-rev-cosmic-rays.pdf

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