Chapter Thirteen

Charged Particle Collisions, Energy Loss, Scattering

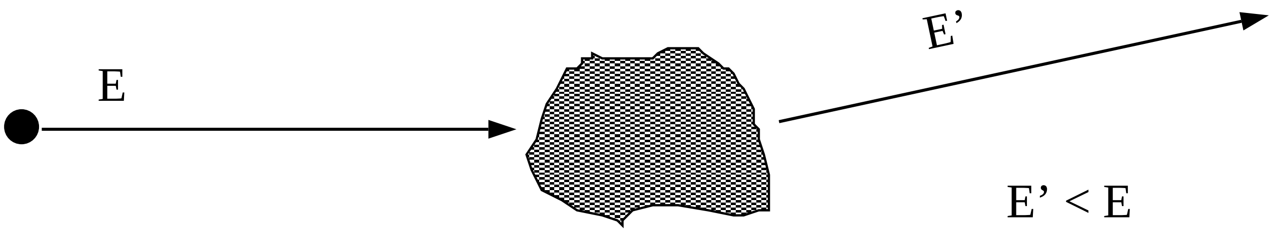
Niels Henrik David Bohr

(1885 - 1962)

September 17, 2001

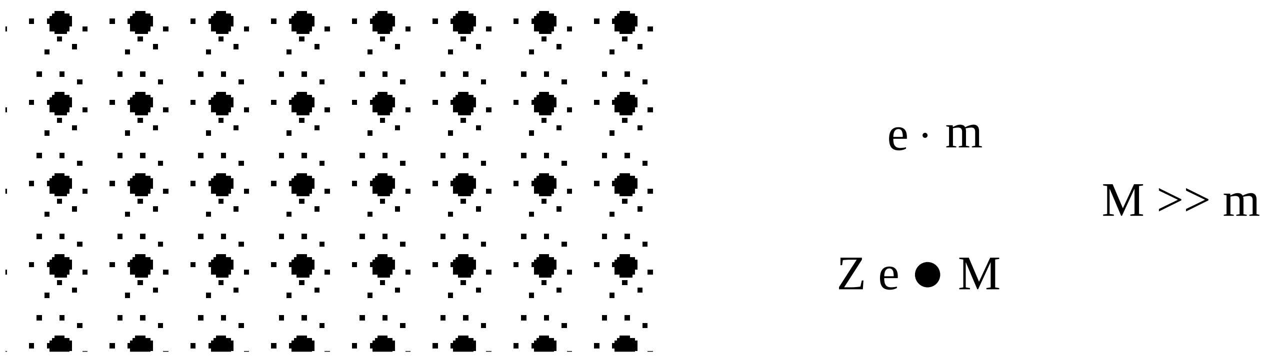
Contents

The topic of interest is a charged particle traversing a material medium. Such a particle looses energy by scattering from the charged particles, electrons and nuclei, in the material.



This is an interesting system from many points of view. Historically it was extremely important in resolving the question of the structure of matter (the Rutherford atom), and at present energy loss is an important phenomenon in particle physics and is also studied in detail by nuclear engineers and by condensed matter physicists in connection with the properties of materials and radiation damage to materials.

The problem can be studied as a straightforward application of electromagnetism; charged particles scatter from one another with the result that energy and momentum are transferred. The scattering centers in a material are of two distinct types; there are electrons of charge $-e$ and small mass $m \sim 10^{-27}\, g$, and there are nuclei of charge Ze with Z up to about 10^2 and large mass $M \sim 10^{-22}\, g$. Thus the nuclear charge is significantly larger than that of an electron, and the nuclear mass is much larger—some 10^5 times larger—than the electronic mass. It is also important to realize that there are Z more electrons than nuclei (Z is the atomic number of the atoms in the material) in a given volume of target material. Consequently the electrons provide Z times as many scattering centers as the nuclei.



As we shall see, it turns out that electrons soak up most of the energy of an incident particle while nuclei are responsible for most of the momentum transfer in the sense that they are more effective than electrons at deflecting the incident particle from its initial direction of motion.

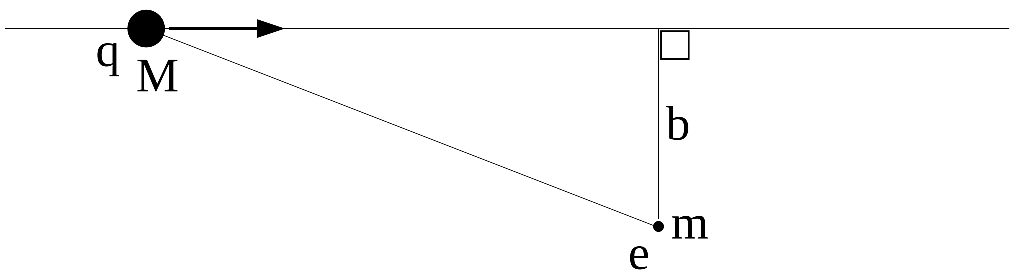
1 Energy Transfer in Coulomb Collisions

The general problem of energy transfer when a charged particle traverses a material is naturally very complicated. We shall approach it a little at a time starting with the classical impulse approximation applied to a pair of particles.

1.1 Classical Impulse Approximation

Consider a particle (q, M), where q is the charge and M, the mass, incident with speed v on a second particle $(-e, m)$ at rest in the frame of our calculation. The incident particle has total energy $M\gamma c^2$, where $\gamma = 1/(1 - v^2/c^2)^{1/2}$. In the impulse approximation the incident particle is treated as undeflected by the collision. Further, the target is approximated as stationary during the collision. Then it is easy to calculate the momentum, or impulse, transferred from the incident particle to the target.

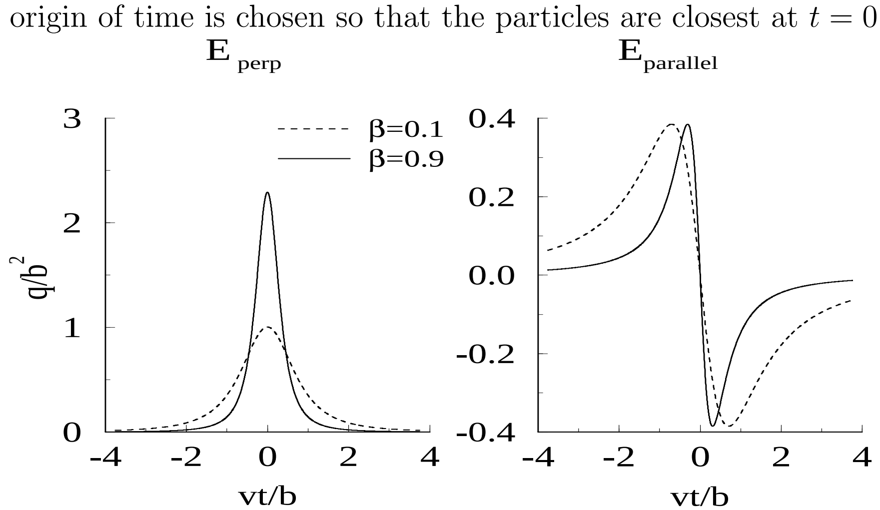
Given the approximation that the incident particle's trajectory is unaffected by the collision, it travels with constant velocity and passes the target at some distance b called the *impact parameter*.



The momentum transferred to the target can be expressed as the integral over time of the force acting on it, and we can find that from knowledge of the electric field produced by the incident particle at the location of the target. From prior calculations in Chapter 11, we know that this field is $\mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_\|$ where the parallel and perpendicular components act parallel and perpendicular to the line of motion of the incident particle. These components are given at the target, by

$$E_\perp(b) = \frac{\gamma q b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \qquad \text{and} \qquad E_\|(b) = -\frac{\gamma q v t}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{1}$$

where the origin of time is chosen so that the particles are closest at $t = 0$.



The integral over time of $E_\|$ is zero while that of $E_\perp$ provides the momentum transferred to the target,

$$
\begin{aligned}
p &= \left|\int_{-\infty}^{\infty} dt\,(-eE_\perp)\right| = \left|\int_{-\infty}^{\infty} dt\,\frac{e\gamma qb}{(b^2+\gamma^2v^2t^2)^{3/2}}\right| = \left|\frac{eqb}{v}\right|\int_{-\infty}^{\infty}\frac{dx}{(b^2+x^2)^{3/2}} \\
&= \left|\frac{qe}{bv}\right|\int_{-\infty}^{\infty}\frac{du}{(1+u^2)^{3/2}} = \frac{2|qe|}{bv}. \qquad (2)
\end{aligned}
$$

Next, we shall assume that $p << mc$ so that the energy transfer to the target may be approximated by the non-relativistic formula $p^2/2m$. This gives us a third approximation whose validity we must scrutinize. Adopting it, we find that the energy transfer to the target is[1]

$$
\Delta E = \frac{p^2}{2m} = \frac{2q^2e^2}{mb^2v^2} = \left(\frac{qe}{b}\right)\frac{(qe/b)}{(mv^2/2)} \propto \frac{e^2}{m}. \qquad (3)
$$

Notice that the energy transfer is proportional to the square of the charge of the target particle and inversely proportional to its mass. Possible targets are electrons and nuclei. A nucleus has a larger charge than an electron by a factor of the atomic number z, giving the nucleus an "advantage" by a factor of z^2 when it comes to extracting energy from the incident particle. However, nuclei are more massive than electrons by a factor of $1836A$ where A is the atomic weight which is as large as or larger than z. Furthermore, there are z more electrons than nuclei to act as targets. *Hence we see that the electrons are more effective than nuclei at taking the energy of the incident particle by a factor of at least 1836.* For this reason, we shall henceforth suppose that the target particle is an electron so long as we are interested in the energy transfer, as opposed to the momentum transfer, from the incident particle to the target.

effect	nucleus	electron
charge	z^2	1
mass	$1/(1836z)$	1
number	1	z
total	$z/(1836)$	z

[1]We shall suppose $qe > 0$ so that the notation is simplified.

1.2 Validity of Approximations

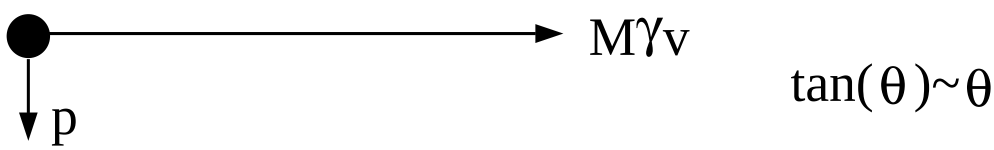
Our simple calculation of the energy transfer contains three distinct approximations.

1. It is assumed that the incident particle is not deflected from its straight-line path. This assumption is valid so long as the actual angle of deflection θ obeys the inequality $\theta << 1$.

2. It is assumed that the target particle is at a particular point during the entire collision. This assumption is valid provided the target recoils a distance d during the collision which is small compared to the impact parameter b, $d << b$.

3. We assumed that the recoiling particle is non-relativistic, $p << mc$.

We can determine the conditions under which the approximations are valid. **First**, the angle of deflection of the incident particle is of order $p/M\gamma v$,



$$\theta \approx \frac{p}{M\gamma v} \approx \frac{2qe}{\gamma b M v^2} = \frac{2}{\gamma}\left(\frac{(qe/b)}{Mv^2}\right). \tag{4}$$

Thus we require that the electrostatic potential energy of interaction at a separation of the impact parameter should be small compared to the energy $M\gamma v^2$ which is something like the energy of the incident particle. This condition is generally met. It is also not independent of the other conditions required for the validity of the impulse approximation.

Second, consider the requirement that the target not recoil far in comparison with b during the collision. From the form of the electric field

$$E_\perp(b) = \frac{\gamma q b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

and hence the force experienced the the particle, we can see that the duration τ of the collision is roughly $b/\gamma v$. During this time the recoil particle moves a distance of order $(p/m)\tau$, assuming it was initially at rest, so the requirement is

$$\frac{b}{\gamma v}\left(\frac{2qe}{mbv}\right) << b \qquad \text{or} \qquad \frac{1}{\gamma}\frac{(qe/b)}{mv^2} << 1. \tag{5}$$

This inequality is much like the previous one but note the replacement of M with the mass m. Given that the target is an electron, which has the smallest mass of all charged particles[2], the present condition is at least as strong as the condition $\theta << 1$. Consequently, we can forget about the latter.

Notice that the condition (5) can also be written as

$$\frac{1}{\gamma}\frac{c^2}{v^2}\frac{r_0}{b} << 1 \tag{6}$$

where $r_0 \equiv e^2/mc^2$ is the *classical radius of the electron* which is about $2.82\times10^{-13}\ cm$. Thus, provided the factor $c^2/v^2\gamma$ is not much larger than unity, this condition is met for impact parameters (almost) down to r_0 which is also about the size of a baryon or nucleus at which point we would expect the calculation to fail for entirely different reasons. Notice, however, that the condition becomes much more severe if v is not large, i.e., if the incident particle is not relativistic. That is not surprising; the collision will last much longer if the incident particle moves slowly and the target has more time to recoil during the collision.

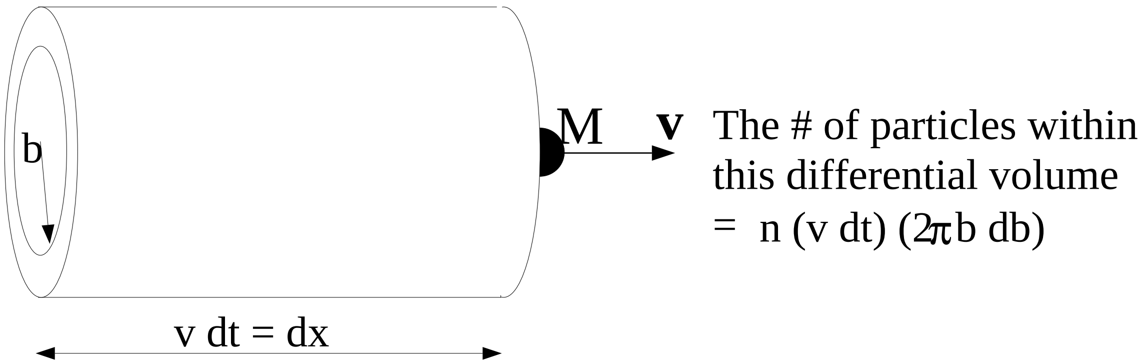
Third, comes the condition that $p << mc$, or $2(qe/b)/mvc << 1$. This condition is not much different from Eq. (5).

1.3 Energy Loss

We have calculated, in the impulse approximation, the energy absorbed from an incident particle by a single electron. There is never just one electron. We have to figure out how to add up the contributions of many electrons to determine how much

[2] As far as anyone knows.

energy the incident particle will lose per unit length of its path. Given an electron density n, then an incident particle having speed v will pass $n(vdt)(2\pi b\,db)$ scatterers per unit time at distances between b and $b + db$.



The consequent energy change of the incident particle in time dt is, from Eq. (3)

$$d^2E = -dtdb\,v2\pi bn\left(\frac{2q^2e^2}{mb^2v^2}\right). \tag{7}$$

Now integrate over b to get the contributions from scatterers at all distances. This must be done with some care. Let's just integrate b from some minimum to some maximum:

$$\frac{dE}{dt} = -\frac{4\pi nq^2e^2}{mv}\int_{b_{min}}^{b_{max}}\frac{db}{b} = -\frac{4\pi nq^2e^2}{mv}\ln(b_{max}/b_{min}). \tag{8}$$

The mathematical necessity of the upper and lower cutoffs on b is clear; the integral would diverge at either end without the cutoff. Physically, what is the reason for them? We have just seen that the impulse approximation breaks down at $b \to 0$ because the recoiling particle recoils by a distance comparable to or larger than the impact parameter during the collision in that limit. Referring back to the condition that our approximation is valid

$$\frac{b}{\gamma v}\left(\frac{2qe}{mbv}\right) << b \quad \text{or} \quad \frac{1}{\gamma}\frac{(qe/b)}{mv^2} << 1\,,$$

we see that a reasonable value for the cutoff is $b_{min} = qe/m\gamma v^2$. This will also make certain that the incident particle's deflection θ is small.

What about the upper cutoff? The physical reason for the breakdown of the impulse approximation (which then necessitates the cutoff) at large b is that when b is large, the collision time $\tau = b/\gamma v$ is long. When this time is long, the natural motions of the target cannot be neglected; that is, the electron or target is not really at rest although we treated it as such when calculating the energy transfer. Most electrons are bound to atoms, molecules, or ions with some binding energy E_e giving them a natural angular frequency of motion $\omega_0 = E_e/\hbar$. The corresponding period is of order $1/\omega_0$. The collision time must be small compared to this time or the impulse approximation, as we have derived it, breaks down. That suggests we choose b_{max} according to $b_{max}/\gamma v = 1/\omega_0$ or $b_{max} = \gamma v/\omega_0$.

Using these cutoffs, we find that the rate of change with time of the incident particle's energy is

$$\frac{dE}{dt} = -\frac{4\pi n q^2 e^2}{mv} \ln\left(\frac{m\gamma^2 v^3}{qe\omega_0}\right). \tag{9}$$

A perhaps more interesting quantity is $dE/dx = v^{-1} dE/dt$,

$$\frac{dE}{dx} = -\frac{4\pi n q^2 e^2}{mv^2} \ln\left(\frac{m\gamma^2 v^3}{qe\omega_0}\right) \tag{10}$$

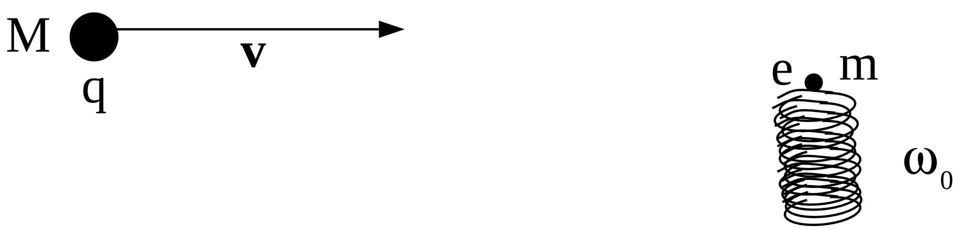
In this derivation, we have determined the lower cutoff on b by looking at the breakdown of the classical impulse approximation. There is also a breakdown associated with quantum effects which implies a somewhat different lower cutoff. The quantum breakdown can be understood by appealing to the uncertainty principle. The value of b is uncertain by an amount related to the momentum of the incident particle. We claim that it has no momentum in the direction in which the impact parameter is measured. We can't really know this to be precisely the case and there has to be an uncertainty in the impact parameter which is of order $\hbar/m\gamma v$. If this uncertainty is comparable to b itself, then our calculation fails. Hence the quantum mechanical cutoff is $b_{min}^{(q)} = \hbar/m\gamma v$. In any given situation, we have to use the larger of the two lower cutoffs. The ratio of the two is

$$\frac{b_{min}^{(q)}}{b_{min}} = \frac{\hbar}{m\gamma v}\frac{\gamma m v^2}{qe} = \frac{\hbar v}{qe} = \frac{1}{\alpha(q/e)}\frac{v}{c} \approx \frac{137}{(q/e)}\frac{v}{c} \tag{11}$$

where $\alpha \equiv e^2/\hbar c \approx 1/137$ is the *fine structure constant.* If this parameter is larger than unity, the quantum cutoff should be employed; if it is smaller than unity, the classical one is appropriate.

2 Collisions with a Harmonically Bound Charge; Energy Loss

One can remove the need for introducing the upper and lower cutoffs on b by doing a more careful treatment of the (classical) energy transfer in the collision. The more careful treatment needed at small b is relegated to the homework. The one needed at large b, which must include the natural motion of the target particle, is given here.



Suppose that the target is bound harmonically at a point, taken as the origin of coordinates, meaning that there is a restoring force $-m\omega_0^2\mathbf{x}$, where $\mathbf{x}$ is the particle's position and ω_0 is the natural frequency of the oscillator, in the absence of damping or perturbing forces. Given that the particle is an electron with mass m and charge $-e$, its equation of motion in the presence of an applied electric field (the one coming from the incident particle) is

$$m\frac{d^2\mathbf{x}}{dt^2} = -m\omega_0^2\mathbf{x} - m\Gamma\frac{d\mathbf{x}}{dt} - e\mathbf{E}(\mathbf{x}, t) \tag{12}$$

where the term $-m\Gamma d\mathbf{x}/dt$ is a damping force proportional to the particle's velocity; Γ is a 'damping constant'. This term is typically small compared to the restoring force term.

To simplify the solution, we will make several **Approximations**.

1. We have not included the magnetic force which acts on the bound charge. This force is smaller than the electric force by a factor of the recoiling particle's velocity divided by c even if the incident particle is relativistic; since the recoiling particle is not relativistic, we may ignore the magnetic force.

2. We make one more approximation which is to evaluate $\mathbf{E}(\mathbf{x}, t)$ at the origin or point where the target particle is bound; this is reasonable provided $b >> |\mathbf{x}|$ because then the electric field will vary but little over distances of order $|\mathbf{x}|$.

We solve Eq. (12) by making a Fourier analysis of the motion. Write

$$\mathbf{E}(t) \equiv \mathbf{E}(0, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \, \mathbf{E}(\omega') e^{-i\omega' t} \tag{13}$$

and

$$\mathbf{x}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \, \mathbf{x}(\omega') e^{-i\omega' t}. \tag{14}$$

The inverse transforms are

$$\mathbf{E}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, \mathbf{E}(t) e^{i\omega t} \tag{15}$$

and

$$\mathbf{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, \mathbf{x}(t) e^{i\omega t}. \tag{16}$$

Substitute Eqs. (13) and (14) directly into the equation of motion and perform the time derivatives to find

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \, \left[-\omega'^2 - i\omega'\Gamma + \omega_0^2 \right] \mathbf{x}(\omega') e^{-i\omega' t} = -\frac{(e/m)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \, \mathbf{E}(\omega') e^{-i\omega' t}. \tag{17}$$

If we multiply by $e^{i\omega t}$ and integrate over t, we obtain a delta-function, $\delta(\omega - \omega')$, and can then integrate trivially over ω' to find a solution for $\mathbf{x}(\omega)$ which is

$$\mathbf{x}(\omega) = -\frac{e\mathbf{E}(\omega)}{m} \frac{1}{\omega_0^2 - i\omega\Gamma - \omega^2}. \tag{18}$$

We could now figure out what is $\mathbf{E}(\omega)$ since we know $\mathbf{E}(t)$ and use it in Eq. (18) to find $\mathbf{x}(\omega)$ and then Fourier transform the latter to find $\mathbf{x}(t)$. But we aren't really interested in $\mathbf{x}(t)$. What we are trying to determine is the energy transferred to the target from the incident charge. That energy can be found as follows:

$$\frac{dE}{dt} = \mathbf{F} \cdot \frac{d\mathbf{x}}{dt} = -e\mathbf{E}(t) \cdot \frac{d\mathbf{x}(t)}{dt} \tag{19}$$

where we again approximate $\mathbf{E}(\mathbf{x}, t)$ with $\mathbf{E}(0, t)$. The total energy transferred in the collision is

$$\begin{aligned}
\Delta E &= -\int_{-\infty}^{\infty} dt\, e\mathbf{E}(t) \cdot \frac{d\mathbf{x}(t)}{dt} \\
&= -\int_{-\infty}^{\infty} dt\, \frac{e}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega'\, \mathbf{E}(\omega') e^{-i\omega' t} \cdot \frac{d}{dt} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \mathbf{x}(\omega) e^{-i\omega t} \right) \\
&= -e \int_{-\infty}^{\infty} d\omega\, (-i\omega) \mathbf{x}(\omega) \cdot \mathbf{E}(-\omega).
\end{aligned} \tag{20}$$

The last step is achieved by, first, taking the time derivative; second, integrating over t to obtain a delta-function $\delta(\omega + \omega')$; and, finally, integrating over ω'.

Because the electric field is real, $\mathbf{E}(-\omega) = \mathbf{E}^*(\omega)$. Similarly, $\mathbf{x}(-\omega) = \mathbf{x}^*(\omega)$; hence

$$\Delta E = ie \int_{-\infty}^{\infty} d\omega\, \omega\, \mathbf{x}(\omega) \cdot \mathbf{E}^*(\omega) = \Re \left[2ie \int_{0}^{\infty} d\omega\, \omega\, \mathbf{x}(\omega) \cdot \mathbf{E}^*(\omega) \right]. \tag{21}$$

Using our solution for $\mathbf{x}(\omega)$, we find

$$\begin{aligned}
\Delta E &= \Re \left(-2i \frac{e^2}{m} \int_0^{\infty} d\omega\, \frac{\omega |\mathbf{E}(\omega)|^2}{\omega_0^2 - i\omega\Gamma - \omega^2} \right) \\
&= \Re \left(-2i \frac{e^2}{m} \int_0^{\infty} d\omega\, \frac{\omega |\mathbf{E}(\omega)|^2 (\omega_0^2 - \omega^2 + i\omega\Gamma)}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2} \right) \\
&= \frac{e^2}{m} \int_0^{\infty} d\omega\, \frac{2\omega^2 \Gamma |\mathbf{E}(\omega)|^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}
\end{aligned} \tag{22}$$

Finally, consider the limit that Γ is very small (small damping). Then the entire weight in the integrand is at $\omega = \omega_0$ which means that the only part of $\mathbf{E}$ which contributes to the energy transfer is the part whose frequency matches the natural frequency of the oscillator. In this limit the integral can be done by evaluating $\mathbf{E}(\omega)$ at ω_0 so that

$$\begin{aligned}\Delta E &\approx \frac{2e^2}{m}|\mathbf{E}(\omega_0)|^2 \int_0^\infty d\omega \, \frac{\omega^2 \Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2} \\ &= \frac{2e^2}{m}|\mathbf{E}(\omega_0)|^2 \int_0^\infty dx \, \frac{x^2}{[(\omega_0/\Gamma)^2 - x^2]^2 + x^2} \end{aligned} \tag{23}$$

where $x \equiv \omega/\Gamma$. The remaining integral is

$$\begin{aligned} I &= \int_0^\infty \frac{dx}{[(\omega_0/\Gamma)^2 - x^2]^2/x^2 + 1} \approx \int_{-\omega_0/\Gamma}^\infty \frac{dy}{(\Gamma/\omega_0)^2[2y\omega_0/\Gamma + y^2]^2 + 1} \\ &\approx \int_{-\infty}^\infty \frac{dy}{4y^2+1} = \frac{\pi}{2}. \end{aligned} \tag{24}$$

Hence

$$\Delta E = \frac{\pi e^2}{m}|\mathbf{E}(\omega_0)|^2 \tag{25}$$

in the limit of $\Gamma << \omega_0$.

We still need to evaluate $\mathbf{E}(\omega_0)$.

If the incident particle is moving in the z direction and the target lies in the x direction relative to the track of the incident particle, then the components of the electric field are

$$\mathbf{E}_\parallel(t) = -\frac{qv\gamma t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}\boldsymbol{\epsilon_3} \quad \text{and} \quad \mathbf{E}_\perp(t) = \frac{\gamma q b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}\boldsymbol{\epsilon_1}. \tag{26}$$

Hence

$$\mathbf{E}_\perp(\omega) = \frac{qb\gamma}{\sqrt{2\pi}} \int_{-\infty}^\infty dt \, \frac{e^{i\omega t}}{(b^2+\gamma^2 v^2 t^2)^{3/2}}\boldsymbol{\epsilon_1} = \frac{qb\gamma}{\sqrt{2\pi}}\frac{b}{\gamma v}\frac{1}{b^3}\int_{-\infty}^\infty dx \, \frac{e^{-i(\omega b/\gamma v)x}}{(1+x^2)^{3/2}}\boldsymbol{\epsilon_1}$$

$$= \frac{q}{bv\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, \frac{e^{-izx}}{(1+x^2)^{3/2}} \boldsymbol{\epsilon_1} = \frac{2q}{bv\sqrt{2\pi}} \int_0^{\infty} dx \, \frac{\cos(zx)}{(1+x^2)^{3/2}} \boldsymbol{\epsilon_1} \tag{27}$$

where $z = \omega b/\gamma v$. The integral we are contemplating is a Bessel function; that is,

$$K_\nu(z) = \frac{2^\nu \Gamma(\nu + 1/2)}{\sqrt{\pi}\, z^\nu} \int_0^\infty dx \, \frac{\cos(xz)}{(1+x^2)^{\nu+1/2}}; \tag{28}$$

specifically,

$$K_1(z) = \frac{2\Gamma(3/2)}{\sqrt{\pi}\, z} \int_0^\infty dx \, \frac{\cos(zx)}{(1+x^2)^{3/2}}. \tag{29}$$

Further, $\Gamma(3/2) = \sqrt{\pi}/2$, so

$$\mathbf{E}_\perp(\omega) = \frac{q}{bv}\sqrt{\frac{2}{\pi}}\, zK_1(z)\boldsymbol{\epsilon_1}. \tag{30}$$

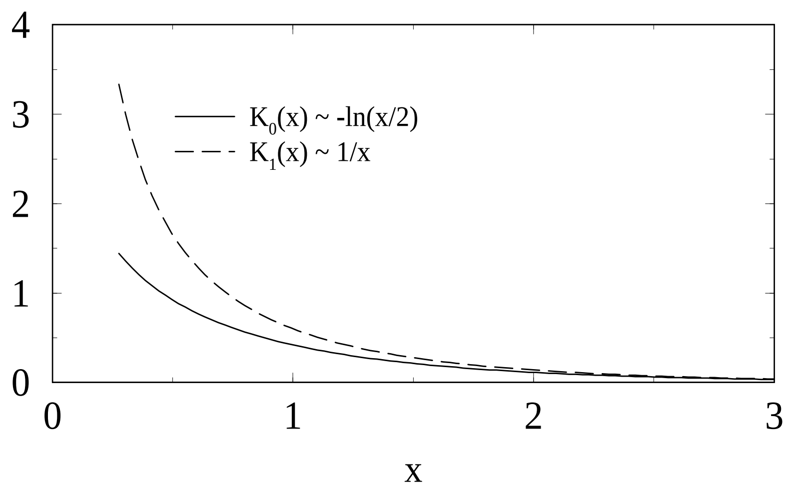
By similar manipulations one finds that

$$\mathbf{E}_\parallel(\omega) = -i\frac{q}{\gamma vb}\sqrt{\frac{2}{\pi}}\, zK_0(z)\boldsymbol{\epsilon_3}. \tag{31}$$

Hence the energy transfer is, from Eq. (25),

$$\Delta E = \frac{\pi e^2}{m}\frac{q^2}{b^2v^2}\frac{2}{\pi}\left[z^2K_1^2(z) + \frac{z^2}{\gamma^2}K_0^2(z)\right] = \frac{2q^2e^2}{mb^2v^2}\left[z^2K_1^2(z) + \frac{z^2}{\gamma^2}K_0^2(z)\right] \tag{32}$$

where $z = \omega_0 b/\gamma v = b/b_{max}$ using $b_{max} = \gamma v/\omega_0$ as per the criterion discussed in the preceding section.



Cutoffs. What is the qualitative behavior of this result as a function of b? For $b << b_{max}$, $z << 1$ and $zK_0(z) \to 0$ as $z \to 0$ while $zK_1(z) \to 1$. Making these substitutions in Eq. (32) we find that for small b, the result is the same as what emerged from the impulse approximation.[3] Thus we must still insert by hand a cutoff for small b (b_{min}). For large $b >> b_{max}$, however, $z >> 1$ and the Bessel functions' behavior is

$$K_0(z) \sim K_1(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \tag{33}$$

so that in this regime of b,

$$\Delta E \approx \frac{2q^2e^2}{mv^2b^2}\left[\frac{\pi}{2z}z^2\left(1+\frac{1}{\gamma^2}\right)e^{-2z}\right] = \frac{q^2e^2\pi z}{mv^2b^2}\left(1+\frac{1}{\gamma^2}\right)e^{-2z}. \tag{34}$$

Thus the large b cutoff is automatically included in this formalism

We can find the total energy loss per unit path length by integrating over b as before. Given an electron density n, then an incident particle traversing a distance dx will pass $n(dx)(2\pi b\, db)$ scatterers with impact parameters between b and $b + db$. The integral for the energy loss by the incident particle can then be put in the form

$$d^2E = -(2\pi b\, db)(v\, dt)n\, \Delta E\,,$$

or, since $b = b_{max} z = \frac{v\gamma}{\omega_0} z$ and $dx = v\, dt$,

$$d^2E = -2\pi\left(\frac{v\gamma}{\omega_0}\right)^2 z\, dz\, dx\, n\Delta E\,.$$

Then, integrating on z, we get

$$\frac{dE}{dx} = -2\pi n\frac{2q^2e^2}{mv^2}\int_{z_{min}}^{\infty}\frac{dz}{z}\left[z^2K_1^2(z) + \frac{z^2}{\gamma^2}K_0^2(z)\right] \tag{35}$$

where $z_{min} = b_{min}/b_{max} = qe\omega_0/m\gamma^2v^3$. The integral, which is

$$I \equiv \int_{z_{min}}^{\infty} dz\, z\left(K_1^2(z) + \frac{1}{\gamma^2}K_0^2(z)\right) = \int_{z_{min}}^{\infty} dz\, z\left(K_1^2(z) + K_0^2(z) - \frac{v^2}{c^2}K_0^2(z)\right), \tag{36}$$

[3]Which is almost miraculous because we approximated $\mathbf{E}(\mathbf{x}, t)$ as $\mathbf{E}(0, t)$ which is not good when b is small; evidently, some cancellation of errors takes place.

can be done by making use of certain identities satisfied by the Bessel functions. These identities are

$$K'_\nu(x) = -K_{\nu-1}(x) - \frac{\nu}{x}K_\nu(x) \qquad \text{and} \qquad K'_\nu(x) = -K_{\nu+1}(x) + \frac{\nu}{x}K_\nu(x); \tag{37}$$

from them, it follows that

$$\begin{aligned} \frac{d}{dx}[xK_0(x)K_1(x)] &= -x[K_0^2(x) + K_1^2(x)] \\ \frac{d}{dx}[x^2(K_1^2(x) - K_0^2(x))] &= -2xK_0^2(x). \end{aligned} \tag{38}$$

It is now easy to do the integral; the result is

$$I = z_{min}K_0(z_{min})K_1(z_{min}) - \frac{v^2}{2c^2}z_{min}^2[K_1^2(z_{min}) - K_0^2(z_{min})]. \tag{39}$$

Now, $z_{min} = qe\omega_0/m\gamma^2v^3 \sim 10^{-7}$ or less for a relativistic particle, so we can expand the Bessel functions in the small argument limit:

$$K_1(x) \approx 1/x \qquad \text{and} \qquad K_0(x) \approx -[\ln(x/2) + 0.577] = \ln(1.123/x). \tag{40}$$

Thus $I = \ln(1.123/z_{min}) - v^2/2c^2$, and

$$\frac{dE}{dx} = -\frac{4\pi nq^2e^2}{mv^2}\left[\ln\left(\frac{1.123m\gamma^2v^3}{qe\omega_0}\right) - \frac{v^2}{2c^2}\right]. \tag{41}$$

This formula may be easily extended to a (slightly) more realistic form, accounting for different charges with different resonant frequencies. Assume an elemental solid with a density of atoms N, each with Z electrons. The Z electrons will be split into groups of f_j electrons distinguished by the resonant frequency of the group ω_j. The oscillator strengths f_j must satisfy the sum rule $\sum_j f_j = Z$. The groups add linearly so that

$$\frac{dE}{dx} = -4\pi nZ\frac{q^2e^2}{mv^2}\left[\ln B_c - \frac{v^2}{2c^2}\right]. \tag{42}$$

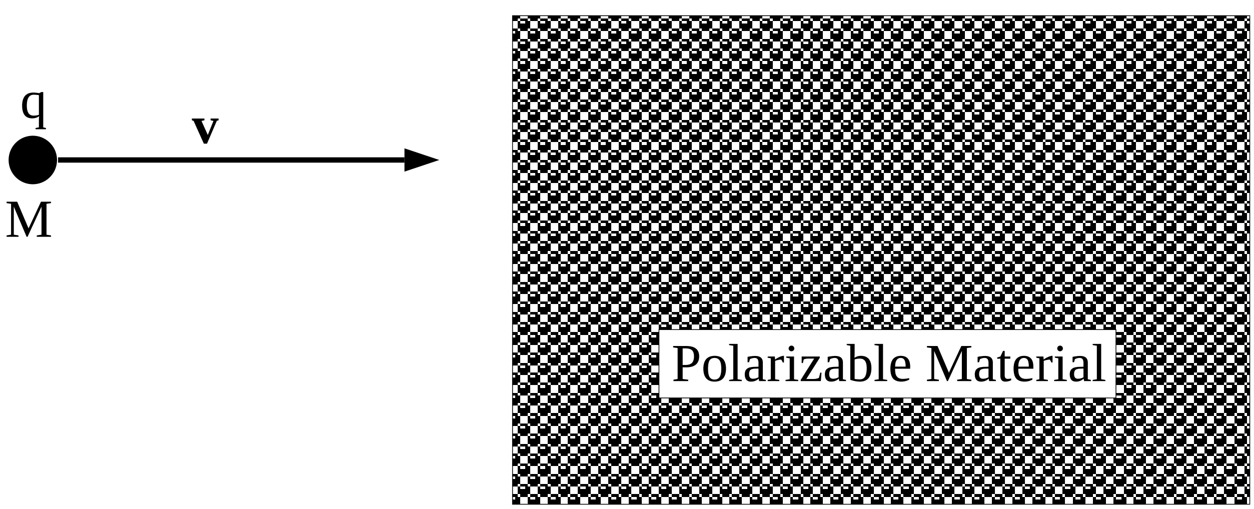
where

$$B_c = \frac{1.123m\gamma^2v^3}{qe < \omega >} \qquad Z\ln < \omega > = \sum_j f_j \ln\omega_j \tag{43}$$

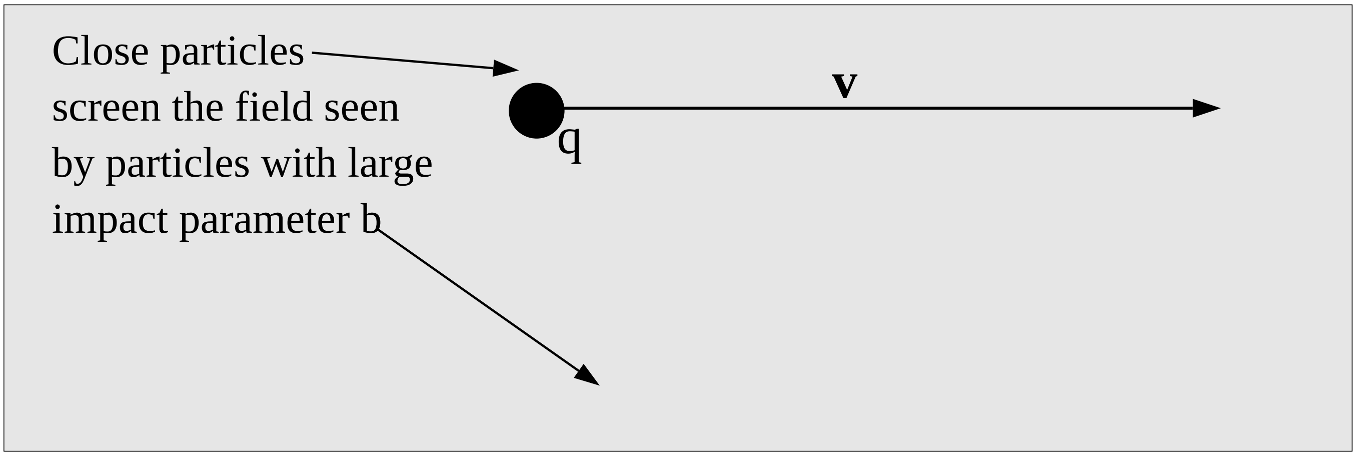
This is the classical energy-loss formula derived by Bohr in 1915. It actually works rather well despite the fact that the effects responsible for the energy loss (scattering of small objects by other small objects) really ought to be treated using quantum theory. The reason why the classical theory works as well as it does is that any macroscopic energy loss is the result of many collisions. The energy loss in each collision is not given very accurately by the classical theory, but Eq. (41) represents the energy loss over a large number of collisions, and that is pretty close to the mark. Thus the usefulness of the classical theory is in part a consequence of statistical effects. Bohr's original formula was eventually superseded by a calculation based on quantum theory and done by Bethe in 1930. Read the appropriate section in Jackson for more details.

3 Density Effect in Energy Loss

A charged particle traversing a material produces a local electric polarization of that material, as a consequence of which the electric field acting on any given charge in the material is not the electric field that we used in the preceding sections.



This "screening" effect is especially important for collisions of large impact parameter b, since then the field will be screened by the charges closer to the path of the incident particle.



Thus the energy loss formulas we derived earlier will overestimate the energy loss of a charged particle traversing a polarizable medium. As we will see, this effect is most important for fast or ultra-relativistic particles.

We can produce a calculation of the consequences of this "screening" effect using the familiar formalism of macroscopic electrodynamics. Let the material have a frequency-dependent dielectric function $\epsilon(\omega)$, as discussed in Chapter 7, so that the displacement and macroscopic electric field, expressed as functions of position and frequency, are related by

$$\mathbf{D}(\mathbf{x},\omega) = \epsilon(\omega)\mathbf{E}(\mathbf{x},\omega); \tag{44}$$

the connection between any field F as a function of $\mathbf{x}$ and ω and the same field as a function of $\mathbf{x}$ and t is

$$F(\mathbf{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, F(\mathbf{x},\omega)e^{-i\omega t} \tag{45}$$

with the inverse transformation

$$F(\mathbf{x},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, F(\mathbf{x},t)e^{i\omega t}. \tag{46}$$

Further, let us introduce Fourier transforms in space:

$$F(\mathbf{x},\omega) = \frac{1}{(\sqrt{2\pi})^3}\int d^3k\, F(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{47}$$

with the inverse

$$F(\mathbf{k},\omega) = \frac{1}{(\sqrt{2\pi})^3}\int d^3x\, F(\mathbf{x},\omega)e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{48}$$

We begin from the macroscopic Maxwell equations with $\mathbf{B} \equiv \mathbf{H}$, i.e., $\mu = 1$; the inhomogeneous equations are

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \qquad \text{and} \qquad \nabla \cdot \mathbf{D} = 4\pi\rho, \tag{49}$$

and the homogeneous field equations may be replace by

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{and} \qquad \mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{50}$$

which ensure that $\nabla \cdot \mathbf{B} = 0$ and that $\nabla \times \mathbf{E} = -\frac{1}{c}\partial\mathbf{B}/\partial t$. Fourier transform the Maxwell equations (49) to find

$$i\mathbf{k} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} - i\frac{\omega}{c}\mathbf{D} \qquad \text{and} \qquad i\mathbf{k} \cdot \mathbf{D} = 4\pi\rho. \tag{51}$$

Similarly, from the Fourier transforms of Eqs. (50) one finds

$$\mathbf{B} = i\mathbf{k} \times \mathbf{A} \qquad \text{and} \qquad \mathbf{E} = -i\mathbf{k}\Phi + i\frac{\omega}{c}\mathbf{A}. \tag{52}$$

Substitution of this pair of equations into the immediately preceding ones and using $\mathbf{D} = \epsilon\mathbf{E}$, we arrive at Fourier transformed wave equations for the potentials:

$$-\mathbf{k}(\mathbf{k}\cdot\mathbf{A}) + k^2\mathbf{A} = \frac{4\pi}{c}\mathbf{J} - \frac{\omega}{c}\epsilon\left[\mathbf{k}\Phi - \frac{\omega}{c}\mathbf{A}\right] \tag{53}$$

and

$$\epsilon\left[k^2\Phi - \frac{\omega}{c}\mathbf{k}\cdot\mathbf{A}\right] = 4\pi\rho. \tag{54}$$

We can make the equations for $\mathbf{A}$ and Φ separate by choosing an appropriate gauge; specifically,

$$\mathbf{k}\cdot\mathbf{A}(\mathbf{k},\omega) = \epsilon\frac{\omega}{c}\Phi(\mathbf{k},\omega)\,, \tag{55}$$

which is a slightly modified form of the Lorentz gauge. Within this gauge, the equations of motion are

$$(k^2 - \epsilon\,\omega^2/c^2)\mathbf{A} = \frac{4\pi}{c}\mathbf{J} \qquad \text{and} \qquad (k^2 - \epsilon\,\omega^2/c^2)\Phi = 4\pi\,\frac{\rho}{\epsilon} \tag{56}$$

which are simple familiar[4] wave equations.

The only macroscopic source is the incident charge[5], so

$$\rho(\mathbf{x}, t) = q\delta(\mathbf{x} - \mathbf{v}t) \qquad \text{and} \qquad \mathbf{J}(\mathbf{x}, t) = \mathbf{v}\rho(\mathbf{x}, t) \tag{57}$$

where we approximate $\mathbf{v}$ as a constant, $\mathbf{v} = v\boldsymbol{\epsilon_3}$. The Fourier transforms of these source densities are

$$\begin{aligned}\rho(\mathbf{k}, \omega) &= \frac{1}{(2\pi)^2}\int d^3x dt\, e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} q\delta(\mathbf{x} - \mathbf{v}t) \\ &= \frac{q}{(2\pi)^2}\int_{-\infty}^{\infty} dt\, e^{-i(\mathbf{k}\cdot\mathbf{v} - \omega)t} = \frac{q}{2\pi}\,\delta(\omega - \mathbf{v}\cdot\mathbf{k})\end{aligned} \tag{58}$$

and, similarly,

$$\mathbf{J}(\mathbf{k}, \omega) = \frac{q\mathbf{v}}{2\pi}\,\delta(\omega - \mathbf{v}\cdot\mathbf{k}). \tag{59}$$

The solutions for the Fourier-transformed potentials are trivially found:

$$\begin{aligned}\Phi(\mathbf{k}, \omega) &= \left(\frac{2q}{\epsilon}\right)\frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v})}{k^2 - \epsilon\,\omega^2/c^2} \\ \mathbf{A}(\mathbf{k}, \omega) &= \left(\frac{2q\mathbf{v}}{c}\right)\frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v})}{k^2 - \epsilon\,\omega^2/c^2}\end{aligned} \tag{60}$$

Now, $\mathbf{E}(\mathbf{k}, \omega) = -i\mathbf{k}\Phi(\mathbf{k}, \omega) + i(\omega/c)\mathbf{A}(\mathbf{k}, \omega)$, so

$$\mathbf{E}(\mathbf{k}, \omega) = 2iq\left(\frac{\omega\mathbf{v}}{c^2} - \frac{\mathbf{k}}{\epsilon}\right)\frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v})}{k^2 - \epsilon\,\omega^2/c^2} \tag{61}$$

and

$$\mathbf{B}(\mathbf{k}, \omega) = i\mathbf{k}\times\mathbf{A}(\mathbf{k}, \omega) = i\left(\frac{2q}{c}\right)(\mathbf{k}\times\mathbf{v})\frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v})}{k^2 - \epsilon\,\omega^2/c^2}. \tag{62}$$

[4]They may not look so familiar because they are in wavenumber and frequency space.

[5]The dielectric function accounts for any sources associated with charges in the material.

Now let us compute the rate at which the incident particle loses energy by finding the flow of electromagnetic energy away from the track of this particle. Let the point at which the fields are to be evaluated be $\mathbf{x} = b\boldsymbol{\epsilon_1}$ and find $\mathbf{E}(\mathbf{x},\omega)$ and $\mathbf{B}(\mathbf{x},\omega)$:

$$\begin{aligned}
\mathbf{B}(\mathbf{x},\omega) &= \frac{1}{(\sqrt{2\pi})^3}\int d^3k\, e^{i\mathbf{k}\cdot\mathbf{x}}\, i\left(\frac{2q}{c}\right)(\mathbf{k}\times\mathbf{v})\frac{\delta(\omega-\mathbf{k}\cdot\mathbf{v})}{k^2-\epsilon\,\omega^2/c^2} \\
&= \frac{(2iq/c)}{(2\pi)^{3/2}}\int d^3k\, e^{ibk_1}(k_2\boldsymbol{\epsilon_1}-k_1\boldsymbol{\epsilon_2})v\frac{\delta(\omega-k_3v)}{k^2-\epsilon\,\omega^2/c^2} \\
&= -\boldsymbol{\epsilon_2}\frac{(2iq/c)}{(2\pi)^{3/2}}\int dk_1dk_2\, k_1e^{ibk_1}\Bigg/\left[k_1^2+k_2^2+\frac{\omega^2}{v^2}\left(1-\epsilon\frac{v^2}{c^2}\right)\right] \qquad (63)
\end{aligned}$$

Set $\lambda^2 = (\omega/v)^2(1-\epsilon\, v^2/c^2)$ and $\beta = v/c$. Then

$$\begin{aligned}
\mathbf{B}(\mathbf{x},\omega) &= -i\boldsymbol{\epsilon_2}\frac{2q}{c(2\pi)^{3/2}}\int dk_1dk_2\frac{k_1e^{ibk_1}}{k_1^2+k_2^2+\lambda^2} = -i\boldsymbol{\epsilon_2}\frac{q}{c\sqrt{2\pi}}\int dk_1\frac{k_1e^{ibk_1}}{\sqrt{k_1^2+\lambda^2}} \\
&= -\boldsymbol{\epsilon_2}\frac{q}{c\sqrt{2\pi}}\frac{d}{db}\left(\int dk_1\,\frac{e^{ibk_1}}{\sqrt{k_1^2+\lambda^2}}\right) = -\boldsymbol{\epsilon_2}\frac{q}{c}\sqrt{\frac{2}{\pi}}\frac{d}{db}\left(\int_0^\infty dx\,\frac{\cos(b\lambda x)}{\sqrt{1+x^2}}\right) \\
&= -\boldsymbol{\epsilon_2}\frac{q}{c}\sqrt{\frac{2}{\pi}}\,\frac{d}{db}[K_0(b\lambda)] = \boldsymbol{\epsilon_2}\frac{q}{c}\sqrt{\frac{2}{\pi}}\,\lambda K_1(b\lambda). \qquad (64)
\end{aligned}$$

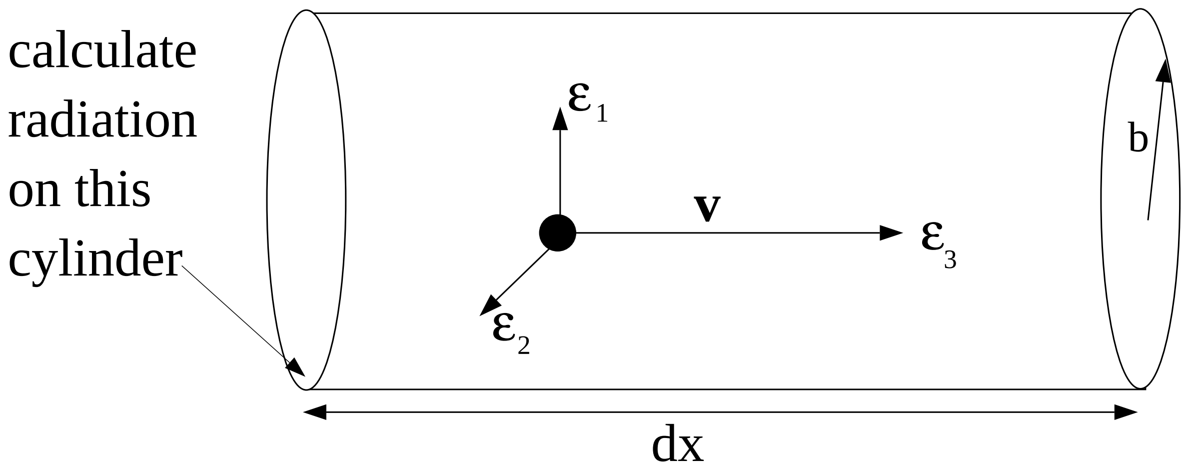
Similarly,

$$\mathbf{E}(\mathbf{x},\omega) = \boldsymbol{\epsilon_1}\frac{q}{v}\sqrt{\frac{2}{\pi}}\,\frac{\lambda}{\epsilon}K_1(b\lambda) - i\boldsymbol{\epsilon_3}\frac{q\omega}{v^2}\sqrt{\frac{2}{\pi}}\left(\frac{1}{\epsilon}-\beta^2\right)K_0(b\lambda). \qquad (65)$$

Next, for real ϵ, λ^2 may be positive or negative depending on whether the incident particle moves more slowly or more rapidly than the speed of light in the medium, $c' = c/\sqrt{\epsilon}$. For $v < c'$, $\lambda^2 > 0$, λ is real, and $\mathbf{E}$ reduces to our previous result except for the appearance of ϵ here and there. It is then a straightforward matter to calculate $\Delta E(b)$ by the same procedure as before, assuming[6] the field acting on a target particle is the same as the macroscopic field.

Rather than reproducing the previous calculation, let's look at an alternative: we shall calculate the radial outward part (ρ component) of the Poynting vector at $\mathbf{x} = \rho\boldsymbol{\rho}$.

[6] A risky assumption.



When this component of $\mathbf{S}$ is integrated over all time and over a closed loop of radius b around the path of the particle, the result is the total electromagnetic field energy which flows away from the particle, per unit length of path, and at distance b from the path. Letting this energy be E_f, to distinguish it from the energy change of anything else (such as the incident particle), we have

$$\left(\frac{dE_f}{dz}\right)_{\rho=b} = \frac{c}{4\pi}2\pi b\int_{-\infty}^{\infty} dt\,(\mathbf{E}\times\mathbf{B})\cdot\mathbf{n} \tag{66}$$

Given the geometry introduced earlier, the quantity $(\mathbf{E}\times\mathbf{B})\cdot\mathbf{n}$ is just $-E_3B_2$.

Let's complete the integral:

$$\begin{aligned}
\left(\frac{dE_f}{dx}\right)_{\rho=b} &= -\frac{cb}{2}\int_{-\infty}^{\infty} dt\,B_2(t)E_3(t) = -\frac{cb}{4\pi}\int_{-\infty}^{\infty} dt d\omega d\omega'\,B_2(\omega')E_3(\omega)e^{-i(\omega+\omega')t} \\
&= -\frac{cb}{2}\int_{-\infty}^{\infty} d\omega B_2(-\omega)E_3(\omega) = -\frac{cb}{2}\int_{-\infty}^{\infty} d\omega E_3(\omega)B_2^*(\omega) \\
&= -cb\Re\left[\int_0^{\infty} d\omega\,B_2^*(\omega)E_3(\omega)\right] \\
&= -\frac{2cbq^2}{\pi v^2}\Re\left[\int_0^{\infty} d\omega\,(-i\omega)\left(\frac{1}{\epsilon}-\beta^2\right)K_0(b\lambda)\frac{1}{c}\lambda^* K_1(b\lambda^*)\right] \\
&= \frac{2q^2}{\pi v^2}\Re\left[\int_0^{\infty} d\omega\,(i\omega\lambda^* b)\left(\frac{1}{\epsilon}-\beta^2\right)K_1(b\lambda^*)K_0(b\lambda)\right],
\end{aligned} \tag{67}$$

an expression first derived by Enrico Fermi.

In order for the integral to have a real part, either λ or ϵ must be complex. If ϵ is real, then λ can still be complex if $\epsilon\beta^2 > 1$ meaning that the particle is travelling faster than the speed of light in the material. In this case one finds the phenomenon of Cherenkov radiation which we shall discuss presently.

For now, let us look at the case of complex ϵ. Introduce the frequency-dependent polarization $\mathbf{P}(\mathbf{x}, \omega)$ via the relation

$$\mathbf{D}(\mathbf{x}, \omega) = \mathbf{E}(\mathbf{x}, \omega) + 4\pi\mathbf{P}(\mathbf{x}, \omega); \tag{68}$$

In a linear medium such as we are considering, $\mathbf{P}(\mathbf{x}, \omega) = \chi(\omega)\mathbf{E}(\mathbf{x}, \omega)$ with $\chi(\omega) = (\epsilon(\omega) - 1)/4\pi$. The frequency dependent polarization is just the Fourier transform in time of the usual polarization $\mathbf{P}(\mathbf{x}, t)$. If we calculate it using the damped harmonic oscillator model introduced above and in chapter 7, we find

$$\mathbf{P}(\omega) = \frac{ne^2}{m} \frac{\mathbf{E}(\omega)}{\omega_0^2 - \omega^2 - i\omega\Gamma} \tag{69}$$

where n is the electron density in the material; the corresponding dielectric function is

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma} \tag{70}$$

where ω_p is the plasma frequency, $\omega_p^2 = 4\pi ne^2/m$.

Now we have an expression for $\epsilon(\omega)$ based on a simple model. We need to do the integral presented in Eq. (67). Unfortunately that cannot be done in terms of simple functions so we shall approximate the integral in a physically reasonable way. The important range of ω should be $\omega \sim \omega_0$ so that $b\lambda \sim b\omega/v \sim b(\omega_0/v) << 1$ for b less than about an atomic size and $v \sim c$; ω_0 is a typical atomic energy. Thus we make the small argument approximations

$$b\lambda^* K_1(b\lambda^*) \approx b\lambda^* \frac{1}{b\lambda^*} = 1 \tag{71}$$

and

$$K_0(b\lambda) \approx \ln(1.123/b\lambda) \tag{72}$$

which leads to

$$\left(\frac{dE_f}{dx}\right)_{\rho=b} = \frac{2q^2}{\pi v^2}\Re\left[\int_0^\infty d\omega\, i\omega \left(\frac{1}{\epsilon} - \beta^2\right) \ln\left(\frac{1.123}{b\lambda}\right)\right]. \tag{73}$$

Because we just fouled up the integrand in the region $\omega >> \omega_0$, we had best make sure that no contribution comes from this region of frequency; physically, we believe this should be the case. Since $\epsilon \to 1$ sufficiently rapidly here (something that should be checked to be sure our belief), we can guarantee convergence of the integral by approximating β^2 with 1. Then

$$\left(\frac{dE_f}{dx}\right)_{\rho=b} = \frac{2q^2}{\pi v^2}\,\Re(I) \tag{74}$$

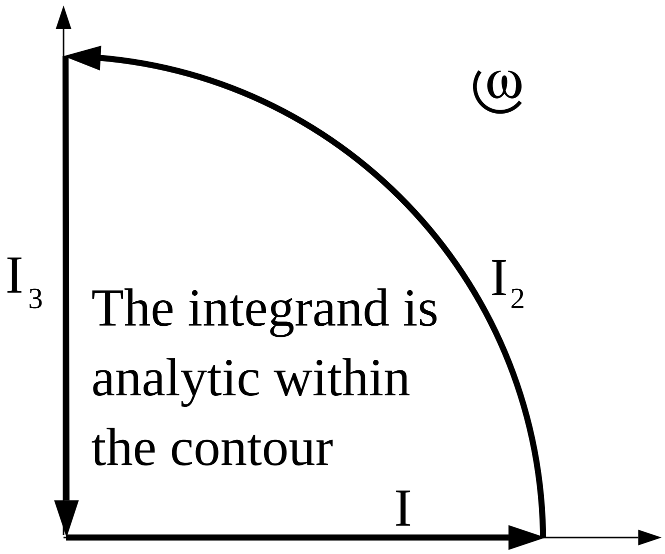
where

$$I = \int_0^\infty d\omega\, i\omega \left[\ln\left(\frac{1.123c}{\omega b}\right) - \frac{1}{2}\ln(1-\epsilon)\right]\left(\frac{1-\epsilon}{\epsilon}\right). \tag{75}$$

Using Eq. (68) for $\epsilon(\omega)$, we have

$$I = i\int_0^\infty d\omega\, \omega \left(\frac{-\omega_p^2}{\omega_0^2 + \omega_p^2 - \omega^2 - i\omega\Gamma}\right)\left[\ln\left(\frac{1.123c}{\omega_p b}\right) - \ln\omega + \frac{1}{2}\ln(\omega^2 - \omega_0^2 + i\omega\Gamma)\right] \tag{76}$$

We can employ the Cauchy theorem to evaluate this integral by closing the contour around the first quadrant; that is, construct a closed path by adding a quarter-circle from a point where ω is large and real to one where it is large and imaginary and then coming down the positive imaginary-ω axis to the origin.



The total integral around this contour is zero because there are no poles of the integrand within it. This point is clarified by looking for the zeroes of the integrand's denominator and by looking for the zeroes of the logarithm's argument. They are located at points in the lower half plane and so are well away from the interior of the contour.

The integral along the imaginary-frequency axis is, with $\omega = i\Omega$, Ω real,

$$
\begin{aligned}
I_3 = -i\int_0^\infty i d\Omega \, \frac{-i\Omega\omega_p^2}{\Omega^2+\omega_0^2+\omega_p^2+\Omega\Gamma}\left[\ln\left(\frac{1.123c}{b\omega_p}\right) - \ln(i\Omega) + \frac{1}{2}\ln[-(\Omega^2+\omega_0^2+\Omega\Gamma)]\right] \\
= i\int_0^\infty d\Omega \, \frac{-\Omega\omega_p^2}{\Omega^2+\omega_0^2+\omega_p^2+\Omega\Gamma}\left[\ln\left(\frac{1.123c}{b\omega_p}\right) - \ln\Omega + \frac{1}{2}\ln(\Omega^2+\omega_0^2+\Omega\Gamma)\right] \quad (77)
\end{aligned}
$$

which is pure imaginary, meaning that $\Re(I_3) = 0$. The integral over the quarter-circle, I_2, is thus such that $-\Re(I_2) = \Re(I)$, or, letting $\omega = \Omega\exp(i\theta)$ on the quarter-circle,

$$
\begin{aligned}
\Re(I) = -\Re\int_0^{\pi/2} i\Omega e^{i\theta} i\Omega e^{i\theta} d\theta \, \left(\frac{-\omega_p^2}{\omega_0^2+\omega_p^2-\Omega^2e^{2i\theta}-i\Omega e^{i\theta}\Gamma}\right) \\
\times\left[\ln\left(\frac{1.123c}{b\omega_p}\right) - \ln\left(\Omega e^{i\theta}\right) + \frac{1}{2}\ln\left(\Omega^2e^{2i\theta}-\omega_0^2+i\Omega\Gamma e^{i\theta}\right)\right] \\
= \omega_p^2\Re\int_0^{\pi/2} d\theta \, \left[\ln\left(\frac{1.123c}{b\omega_p}\right) + \mathcal{O}\left(\frac{\Gamma}{\Omega}\right)\right] = \omega_p^2\frac{\pi}{2}\ln\left(\frac{1.123c}{b\omega_p}\right). \quad (78)
\end{aligned}
$$

Hence,

$$\left(\frac{dE_f}{dz}\right)_{\rho=b} = \left(\frac{q^2\omega_p^2}{c^2}\right)\ln\left(\frac{1.123c}{b\omega_p}\right); \tag{79}$$

The negative of this quantity is the energy loss of the incident particle per unit distance traveled.

This result is to be compared with the one we found before taking screening into account,

$$\left(\frac{dE}{dz}\right)_{\rho>b} = -\frac{q^2\omega_p^2}{c^2}\left[\ln\left(\frac{1.123\gamma c}{b\omega_0}\right) - \frac{1}{2}\right]. \tag{80}$$

The two differ significantly in principle, if not numerically. In particular, the dependence of our original formula on the specific natural frequency of the target, ω_0, is gone, replace by a dependence on ω_p which depends only on the density of the target electrons. Also, a factor of γ has, in our most recent result, disappeared from the argument of the logarithm, meaning that the energy loss by highly relativistic charged particles is much reduced by the screening effect.

4 Cherenkov Radiation

We are also in a position to calculate energy loss by Cherenkov radiation which is something that takes place when the incident particle's speed exceeds the speed of light in the medium. We can avoid the mechanism just discussed and so isolate the Cherenkov radiation mechanism by letting ϵ be real (no damping). In this approximation we will also miss the attenuation of the radiation. Under these conditions, and as discussed in the last section, the only way to get any radiation is if

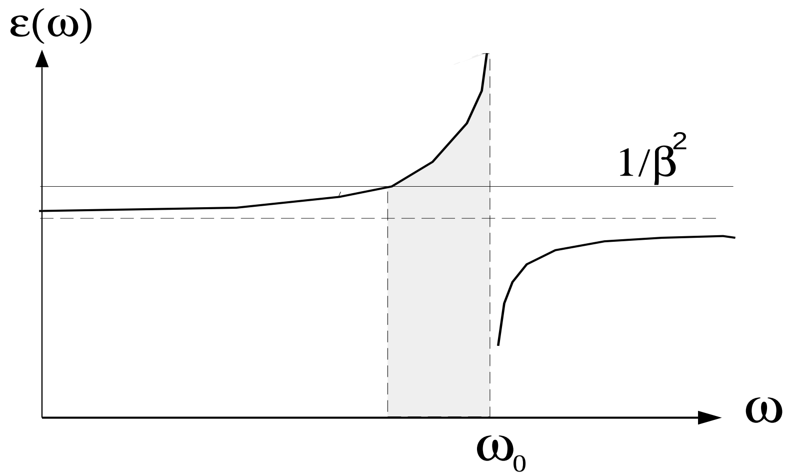
$$\lambda = \frac{\omega}{v}\sqrt{1-\epsilon(\omega)\beta^2} \in \mathrm{C}\,, \tag{81}$$

or, more correctly, λ must be imaginary. We must have $v^2 > c^2/\epsilon$ or there will be no radiation. Since $n = \sqrt{\epsilon}$, then $c/\sqrt{\epsilon}$ is the speed of light in the medium, and thus the condition for radiation is that the particle exceed the speed of light in the medium.

This will not happen for all frequencies. By assuming a simple model dielectric function

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} \,, \tag{82}$$

and expressing the condition as $\epsilon(\omega) > 1/\beta^2$ we can see that the radiation tends to be emitted near regions of anomalous dispersion.



Under these conditions, we evaluate the fields which are present at distance b from the axis of the incident particle, using b large enough that we can make simple approximations to the Bessel functions, $b|\lambda| >> 1$. Then

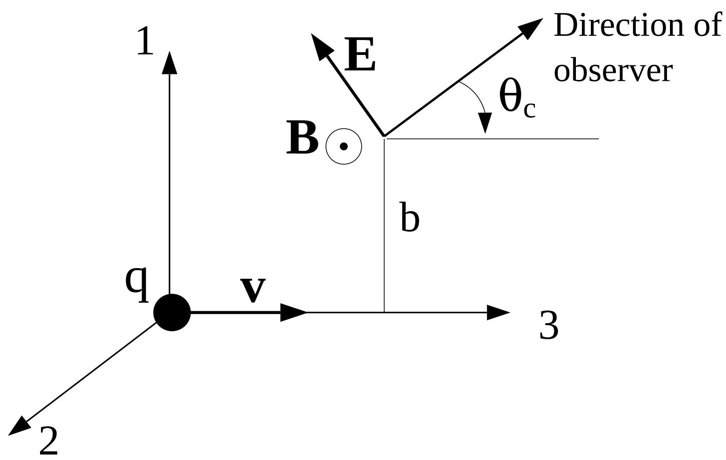
$$K_0(\lambda b) \approx K_1(\lambda b) \approx \sqrt{\frac{\pi}{2\lambda b}} e^{-\lambda b} \tag{83}$$

and so,

$$\mathbf{B}(\mathbf{x}, \omega) = \boldsymbol{\epsilon_2} \frac{q}{c} \sqrt{\frac{\lambda}{b}}\, e^{-\lambda b}\,. \tag{84}$$

Similarly,

$$\mathbf{E}(\mathbf{x}, \omega) = \boldsymbol{\epsilon_1} \frac{q}{\epsilon v} \sqrt{\frac{\lambda}{b}}\, e^{-\lambda b} - i\boldsymbol{\epsilon_3} \frac{q\omega}{v^2} \frac{1}{\sqrt{\lambda b}} \left(\frac{1}{\epsilon} - \beta^2\right) e^{-\lambda b}. \tag{85}$$



Then from Eq. (67) and the equations above, the field energy passing through the cylinder of radius b per unit length is

$$\left(\frac{dE}{dz}\right)_C = \frac{2q^2}{\pi v^2}\Re\left[\int_0^\infty d\omega\, i\omega\frac{\pi}{2}\sqrt{\frac{\lambda^*}{\lambda}}e^{-(\lambda+\lambda^*)a}\left(\frac{1}{\epsilon}-\beta^2\right)\right]. \tag{86}$$

The wonderful thing that happens when λ is pure imaginary is that the exponential functions have imaginary arguments and will not become small as b becomes large. Thus we find the energy given off as *Cherenkov radiation* to be

$$\left(\frac{dE}{dz}\right)_C = \frac{q^2}{v^2}\Re\left[\int d\omega\, i\omega\sqrt{-1}\left(\frac{1-\epsilon\beta^2}{\epsilon}\right)\right] = \frac{q^2}{c^2}\int d\omega\,\omega\left(1-\frac{1}{\epsilon\beta^2}\right) \tag{87}$$

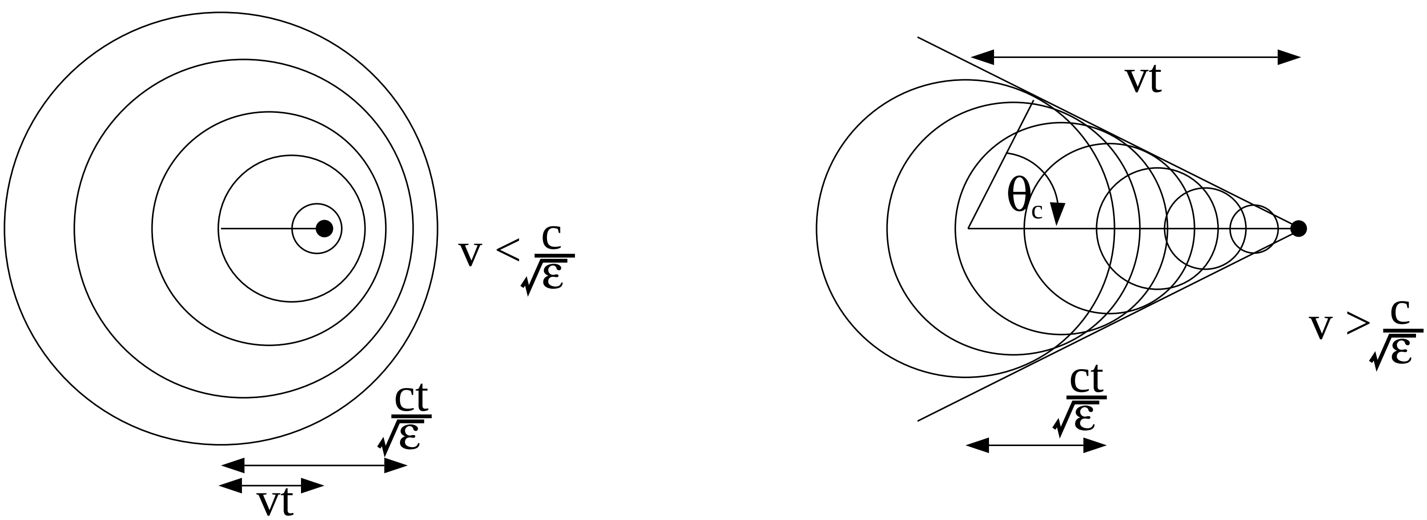
where the integration extends over only those frequencies $\epsilon\beta^2 > 1$. One can see that this is indeed radiative energy loss because it is independent of b provided only b is large enough that the Bessel functions are well-represented by their large-argument forms. In this respect it is quite distinct from the energy loss by transfer of energy to other charged particles that we studied earlier (real as opposed to virtual photons). We were able to treat that energy loss by examining the energy carried by the electromagnetic fields because the mechanism by which the energy is transferred from one particle to another is by means of the fields; in effect, we did that calculation in such a way as to "intercept" the energy that was on its way from one charge to another.

From the picture above it is clear that the radiation is completely linearly polarized in the plane containing the observer and the path of the particle. In addition the angle θ_c of emission of Cherenkov radiation relative to the direction $\boldsymbol{\epsilon_3}$ of the particle's velocity is given by

$$\cos(\theta_c) = \frac{E_1}{\sqrt{E_1^2 + E_3^2}} = \frac{c/n}{v} \tag{88}$$

where $n = \sqrt{\epsilon}$. Thus the condition that λ be complex, and thus that required for Cherenkov radiation, can be rephrased as the requirement that θ_c be a physical angle with a cosine less than unity.

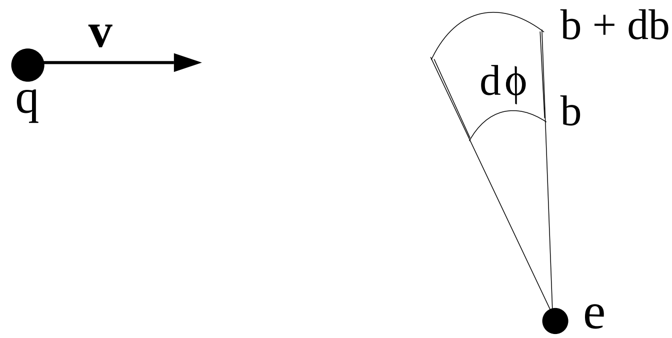
As shown in the picture below, the emission angle θ_c can also be interpreted in terms of a shock wave angle.



5 Momentum Transfer

The final topic we shall study in this chapter is the deflection of the incident particle produced by scattering from the particles in the material through which it moves. The targets mainly responsible for the deflection turn out to be the highly charged ones—the nuclei.

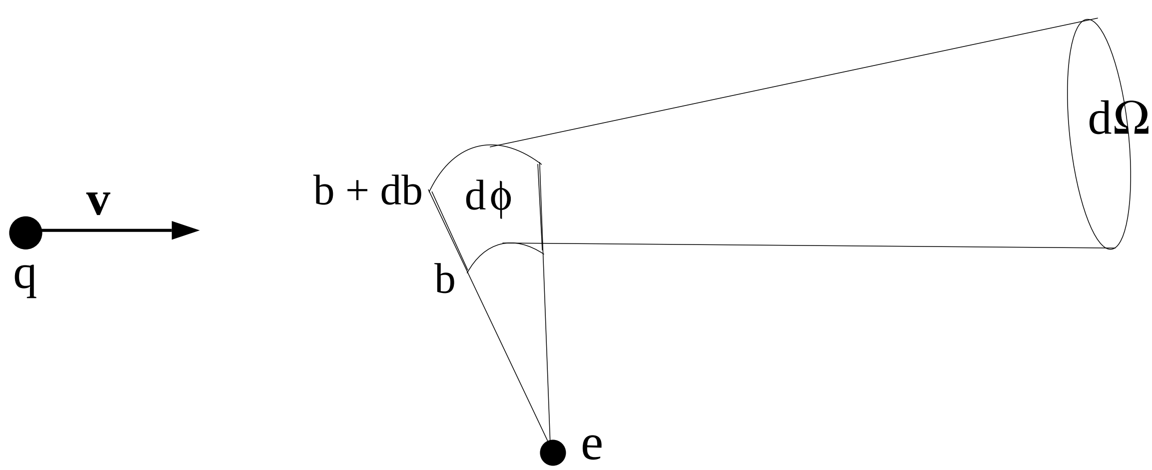
We start by introducing the number of particles incident per unit time on the target with an impact parameter between b and $b + db$ and at an azimuthal angle between ϕ and $\phi + d\phi$.



If the incident beam has a particle number density n and a speed v, then the incident flux is nv particles per unit area per unit time, and the number incident in the area element just described is

$$d^2N = nvbdbd\phi. \tag{89}$$

Now, given a smoothly varying scattering potential, these particles will, after scattering, show up in some element of solid angle $d\Omega$.



Hence we can write that

$$d^2N = N'd\Omega \tag{90}$$

where N' is the number of particles scattered into unit solid angle in unit time and $d\Omega$ is the element of solid angle into which the particular d^2N particles under consideration are scattered. Using Eq. (89), we have

$$nvbd\phi db = N'd\Omega \qquad \text{or} \qquad bd\phi db = \frac{N'}{nv}d\Omega \tag{91}$$

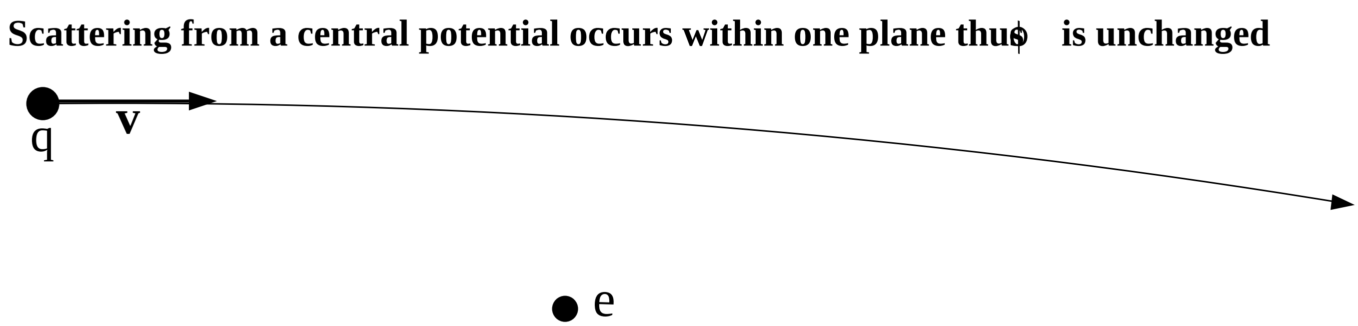
The quantity N' is proportional to the incident particle flux; that is, the number of particles per unit solid angle that come out in some given direction is directly proportional to the incident flux. Hence a more intrinsic measure of the scattering than N' is provided by the quantity N'/nv, the *differential scattering cross-section* $d\sigma/d\Omega$:

$$\frac{d\sigma}{d\Omega} \equiv \frac{N'}{nv} \tag{92}$$

Making this substitution in Eq. (91), we get

$$bd\phi db = \frac{d\sigma}{d\Omega}d\phi \sin\theta d\theta \tag{93}$$

We will also assume that the potential between the incident particle and the scatterer is central. In this case we have azimuthal symmetry so the particles incident on the target in some increment $d\phi$ of azimuthal angle around ϕ are scattered into the same element of azimuthal angle,



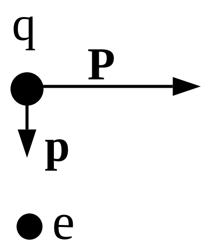
thus we find

$$b\,db = \frac{d\sigma}{d\Omega}\sin\theta d\theta \qquad \text{or} \qquad \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{94}$$

where θ is the angle by which the particle is deflected or scattered.

The differential scattering cross-section, by its definition, has dimensions of length squared or area. We can evaluate it if we have an equation relating b and θ. In the impulse approximation, the scattering angle θ is given by ratio of the momentum transfer to the incident momentum; and that is, from Eq. (2),

$$|\theta| = \frac{p}{P} = \left|\frac{2qe}{Pvb}\right| \tag{95}$$



where, in this equation, $\mathbf{P} = \gamma M\mathbf{v}$ is the momentum of the incident particle, and $p = \frac{2|qe|}{bv}$ (Eq. (2)) is the momentum transfer from the incident particle to the target. From this relation we can evaluate $|d\theta/db|$ and find that the cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{Pvb^2}{2qe}\right| = \frac{Pv}{2qe\sin\theta}\left(\frac{2qe}{Pv\theta}\right)^3 = \left(\frac{2qe}{Pv}\right)^2\frac{1}{\theta^4} \tag{96}$$

where we make the small angle approximation $\theta \approx \sin\theta$ which is valid anywhere that the impulse approximation is valid. In this, the small-angle regime, our result matches the Rutherford scattering cross-section.

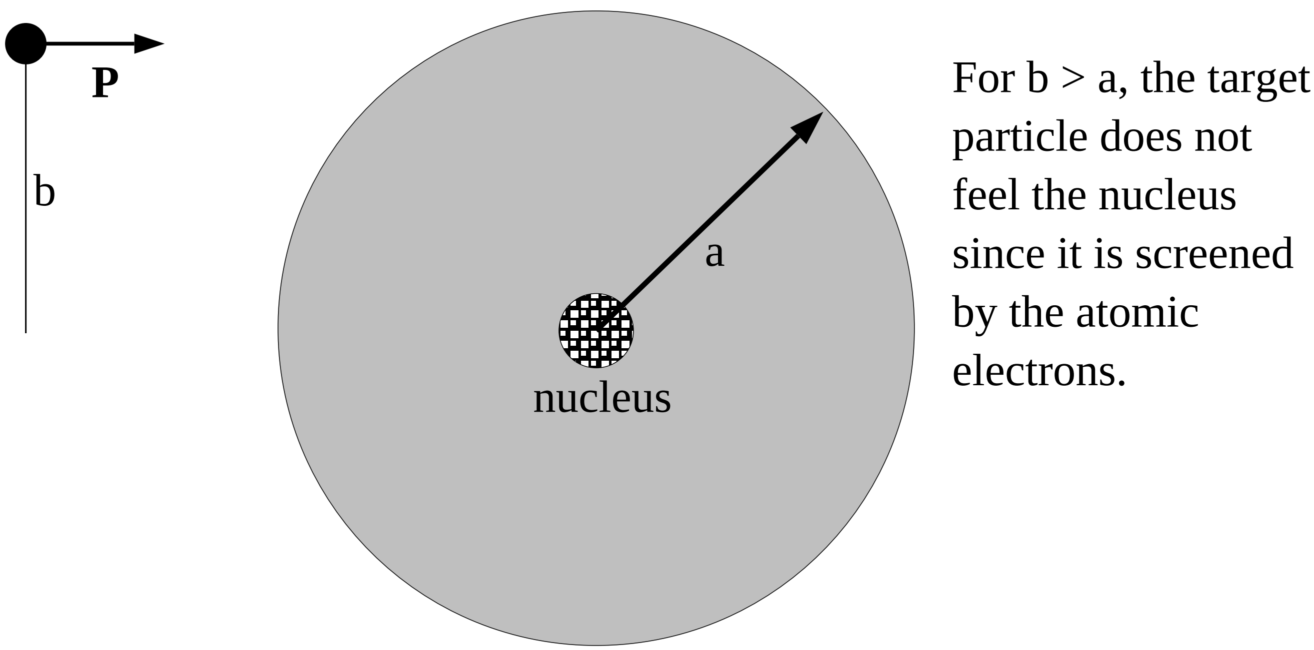
From Eq. (96) we can see that nuclei are more effective than electrons at producing a given deflection θ. The charge e that appears in the cross-section is the charge of the target, a holdover from when we let the target be an electron. More generally, replace this charge by ze, in case the target is, *e.g.,* a nucleus.

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{Pvb^2}{2qze}\right| = \frac{Pv}{2qze\sin\theta}\left(\frac{2qze}{Pv\theta}\right)^3 = \left(\frac{2qze}{Pv}\right)^2\frac{1}{\theta^4} \tag{97}$$

One can then see that the cross-section is proportional to z^2, meaning that a nucleus is more effective by a factor of z^2 at producing a given angle of deflection θ. At the same time there are z times as many electrons, leading to z times as many scattering events. This is not enough to offset the larger cross-section produced by the nuclei, and therefore they are the dominant scatterers where deflection of the incident particle is concerned.

5.1 Average Angle of Deflection

Of course the target is rarely composed of a single atom. Rather, we generally scatter from a molecular solid, or material. Here, we want to calculate a typical or average angle of deflection produced in a scattering event. That will require integrating over θ using $d\sigma/d\Omega$ as the distribution function. Cutoffs on the integration must be introduced. At small θ, corresponding to large b, the cutoff is determined by the condition $b_{max} \sim a$ where a is an atomic size.



The reason is that for $b > a$, the incident particle passes completely outside of the electronic shell surrounding the nucleus and so the interaction between the incident particle and nucleus is almost completely screened. Thus

$$\theta_{min} \approx \left| \frac{qze}{Pvb_{max}} \right| \approx \left| \frac{qze}{Pva} \right|. \tag{98}$$

This still leaves a large range of impact parameter b, since the nuclear radius $\sim 10^{-13}$ cm. and that of a typical atomic radius is $\sim 10^{-8}$ cm.. An alternative, quantum-based argument can be made for choosing $\theta_{min} \sim \hbar/pa$. There is also a maximum scattering angle which is not of much significance in the present context; we may suppose that θ_{max} is of order one.

Given appropriate cutoffs, we can determine the mean value of θ^2 in scattering events. Using the small-angle approximation for all trigonometric functions, we have

$$< \theta^2 >= \frac{\int d\Omega \, \theta^2 [d\sigma/d\Omega]}{\int d\Omega \, [d\sigma/d\Omega]} \approx \frac{\int_{\theta_{min}}^{\theta_{max}} d\theta/\theta}{\int_{\theta_{min}}^{\theta_{max}} d\theta/\theta^3}$$

$$= \frac{2\ln(\theta_{max}/\theta_{min})}{1/\theta_{min}^2 - 1/\theta_{max}^2} \approx 2\theta_{min}^2 \ln(\theta_{max}/\theta_{min}). \tag{99}$$

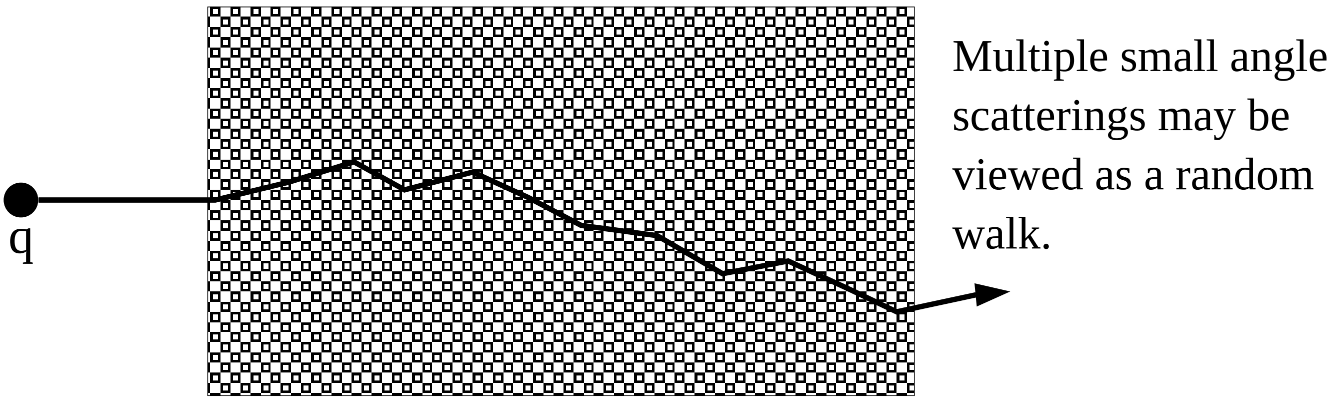
This result is some not-very-large multiple[7] of θ_{min}^2. Hence, a single scattering event cannot be expected to deflect the incident particle very much.

A sizable net deflection can be obtained in two quite different ways. One is that a large number of small-angle scatterings can result in a large deflection. The other is that a single large-angle scattering, though rare, can occur. If one bombards a thin slab of target material with a beam of particles, then what one finds is that most of the particles which come through will have experienced a large number of small-angle scatterings and no large-angle scatterings. These will have a distribution of net scattering angles which reflects their experience (many small-angle scatterings). Some particles, however, will have experienced a large-angle scattering in addition to the many small-angle scatterings. They will have a distribution of scattering angles which reflects their experience and which will be quite unlike the distribution of the particles which experience only small-angle scatterings. Let's give each of these possibilities a little further thought.

[7]Because the cross-section is strongly peaked at small angles.

5.1.1 Distribution of Small Angle Scattering

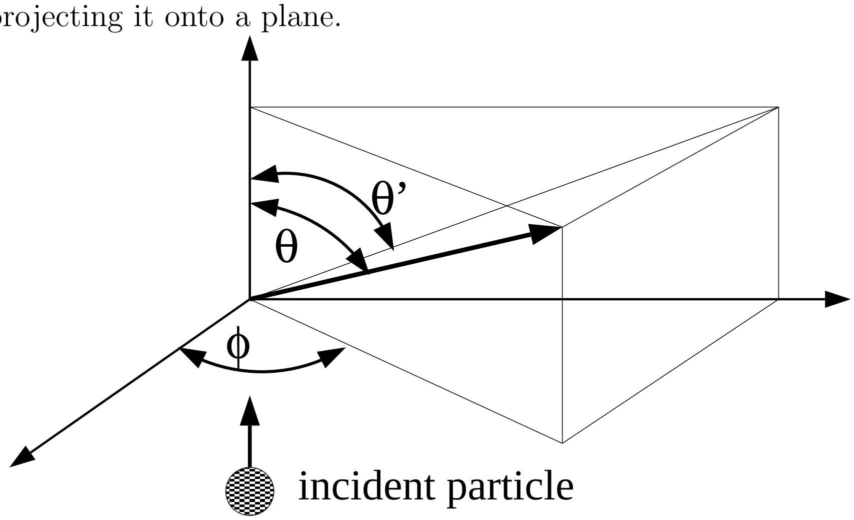
If the particle experiences only a large number of small-angle scattering events, its deflection will resemble a random walk.



A collection of such random walkers will provide a distribution of observed scattering angles which will have approximately a Gaussian form,

$$P(\theta) \sim e^{-(\theta^2/<\Theta^2>)}, \tag{100}$$

where $< \Theta^2 >$ is the width of the distribution. To carry the analysis further in a quantitative manner, let's make the random walk effectively one-dimensional by projecting it onto a plane.



Consider a particle that is scattered into the direction (θ, ϕ); project this direction onto the y-z plane where it becomes θ' with $\theta' = \theta \sin\phi$ for $\theta << 1$. Hence $\theta'^2 =$

$\theta^2 \sin^2 \phi$, and the observed mean value of θ'^2 in single scattering events is

$$< \theta'^2 >= \frac{\int d\Omega \, \theta'^2(\theta, \phi)[d\sigma/d\Omega]}{\int d\Omega \, [d\sigma/d\Omega]} = \frac{1}{2} < \theta^2 > . \tag{101}$$

Also, $< \theta' >= 0$. Assuming that the scattering directions produced by the different collisions that any one particle suffers are independent, and that there are many such collisions, then, from the theory of the elementary one-dimensional random walk, the normalized distribution of observed net scattering angles θ' is well-approximated by a Gaussian

$$P_m(\theta') = \frac{1}{\sqrt{\pi < \Theta^2 >}} e^{-\theta'^2 / < \Theta^2 >} \tag{102}$$

with the random walk distribution width

$$< \Theta^2 >= N < \theta'^2 >, \tag{103}$$

where N is the mean number of collisions experienced by each particle in traversing the material. If the total cross-section is σ, the density of scatterers is n, and the thickness of the slab is a, then $N = n\sigma a$ and so

$$< \Theta^2 >= n\sigma a < \theta'^2 > . \tag{104}$$

For our particular cross-section Eq. (97), $\sigma = \pi(2qze/Pv)^2/\theta_{min}^2$, so, using also Eq. (99),

$$< \Theta^2 >= 2\pi n \left(\frac{2qze}{Pv} \right)^2 a \ln(\theta_{max}/\theta_{min}). \tag{105}$$

5.1.2 The Distribution of Large Angle Scattering

This distribution may be contrasted with the one that arises for particles which undergo a single large-angle scattering and many small-angle ones. If the net effect of the latter is less than the deflection produced by the former, which in some sense defines what we mean by a large-angle scattering, then we need only consider the distribution produced by a single large-angle event. The number of such events is proportional to the cross-section or, for[8] $\theta << 1$,

$$d\sigma = \frac{d\sigma}{d\Omega}d\Omega = \left(\frac{2qze}{pv}\right)^2 \frac{1}{\theta^4} d\phi\, \theta d\theta. \tag{106}$$

We may convert θ to θ' using $\theta = \theta'/\sin\phi$,

$$d\sigma = \left(\frac{2qze}{pv}\right)^2 \frac{d\theta'}{\theta'^3} \sin^2\phi\, d\phi. \tag{107}$$

Now integrate ϕ from zero to π to pick up all events corresponding to $\theta' > 0$. The result is that[9]

$$d\sigma = \frac{\pi}{2}\left(\frac{2qze}{pv}\right)^2 \frac{d\theta'}{\theta'^3}. \tag{108}$$

For a slab of thickness a with a density n of scatterers, the probability of having a single large-angle scattering in an interval $d\theta'$ around θ' is

$$P_s(\theta')d\theta' = nad\sigma = \frac{\pi}{2}na\left(\frac{2qze}{pv}\right)^2 \frac{d\theta'}{\theta'^3}. \tag{109}$$

Because this distribution falls off only as θ'^{-3} while the multiple-scattering distribution falls off exponentially as θ'^2, there is some angle θ_0 such that the single-scattering distribution is larger than the multiple scattering one for $\theta' > \theta_0$ and conversely.

Roughly speaking, the total distribution of scattered particles as a function of θ' is just P_m for $\theta' < \theta_0$ and P_s for $\theta' > \theta_0$. In any given system, one can easily compute the two distributions along with θ_0. It is expected that the description will work quite well for θ' significantly smaller than θ_0 and also for θ' significantly larger. For $\theta' \approx \theta_0$, the actual behavior is complicated considerably by the contribution of particles that have undergone several scatterings through "almost-large" angles. There are not enough such scattering events per particle for them to be properly treated using statistical methods, and they are not easily treated in any other way, except for numerical simulations.

---

[8] Evidently, "large-angle" means an angle large compared to θ_{min}; it does not mean an angle so large as to be of order one.

[9] The original $d\sigma$ is a second-order differential; the result of integrating over ϕ, unfortunately still called $d\sigma$, is a first-order differential.