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Collisions and Scattering Theory

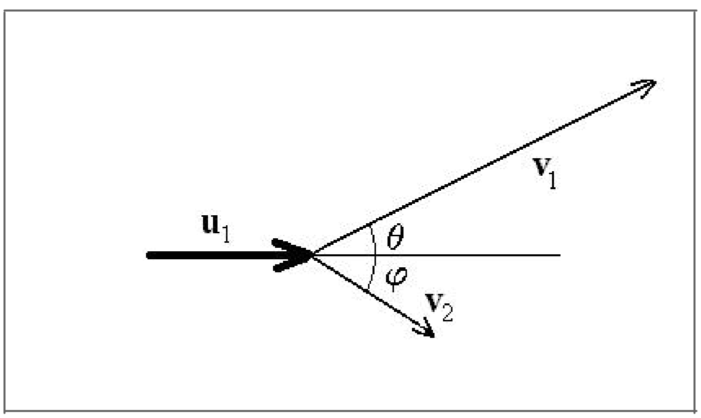
1 Two-Particle Collisions in the LAB Frame

Consider the collision of two particles (labeled 1 and 2) of masses m_1 and m_2, respectively. Let us denote the velocities of particles 1 and 2 **before** the collision as $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, while the velocities **after** the collision are denoted $\mathbf{v}_1$ and $\mathbf{v}_2$. Furthermore, the particle momenta before and after the collision are denoted $\mathbf{p}$ and $\mathbf{q}$, respectively.

To simplify the analysis, we define the laboratory (LAB) frame to correspond to the reference frame in which m_2 is at rest (i.e., $\mathbf{u}_2 = 0$); in this collision scenario, m_1 acts as the **projectile** particle and m_2 is the **target** particle. We now write the velocities $\mathbf{u}_1$, $\mathbf{v}_1$, and $\mathbf{v}_2$ as

$$\left. \begin{array}{rcl} \mathbf{u}_1 & = & u\,\widehat{\mathbf{x}} \\ \mathbf{v}_1 & = & v_1\,(\cos\theta\,\widehat{\mathbf{x}} \;+\; \sin\theta\,\widehat{\mathbf{y}}) \\ \mathbf{v}_2 & = & v_2\,(\cos\varphi\,\widehat{\mathbf{x}} \;-\; \sin\varphi\,\widehat{\mathbf{y}}) \end{array} \right\}, \tag{1}$$

where the **deflection** angle θ and the **recoil** angle φ are defined in the Figure below.



The conservation laws of momentum and energy

$$m_1\mathbf{u}_1 \;=\; m_1\mathbf{v}_1 \;+\; m_2\mathbf{v}_2 \quad\text{and}\quad \frac{m_1}{2}\,u^2 \;=\; \frac{m_1}{2}\,|\mathbf{v}_1|^2 \;+\; \frac{m_2}{2}\,|\mathbf{v}_2|^2$$

can be written as

$$\alpha\,(u \;-\; v_1\,\cos\theta) \;=\; v_2\,\cos\varphi, \tag{2}$$

$$\alpha\,v_1\,\sin\theta \;=\; v_2\,\sin\varphi, \tag{3}$$

$$\alpha\,(u^2 \;-\; v_1^2) \;=\; v_2^2, \tag{4}$$

where $\alpha = m_1/m_2$ is the ratio of the projectile mass to the target mass.

Since the **three** equations (2)-(4) are expressed in terms of **four** unknown quantities $(v_1, \theta,\ v_2, \varphi)$, for given incident velocity u and mass ratio α, we must choose one angle as an independent variable. Here, we choose the recoil angle φ of the target particle, and proceed with finding expressions for $v_1(u, \varphi; \alpha)$, $v_2(u, \varphi; \alpha)$ and $\theta(u, \varphi; \alpha)$. First, using the square of the mometum components (2) and (3), we obtain

$$\alpha^2\, v_1^2 \;=\; \alpha^2\, u^2 \;-\; 2\,\alpha\, u v_2\, \cos\varphi \;+\; v_2^2. \tag{5}$$

Next, using the energy equation (4), we find

$$\alpha^2\, v_1^2 \;=\; \alpha\, \left(\alpha\, u^2 \;-\; v_2^2\right) \;=\; \alpha^2\, u^2 \;-\; \alpha\, v_2^2, \tag{6}$$

so that these two equations combine to give

$$v_2(u, \varphi; \alpha) \;=\; 2 \left(\frac{\alpha}{1+\alpha} \right) u\, \cos\varphi. \tag{7}$$

Once $v_2(u, \varphi; \alpha)$ is known and after substituting Eq. (7) into Eq. (6), we find

$$v_1(u, \varphi; \alpha) \;=\; u\, \sqrt{1 \;-\; 4\, \frac{\mu}{M}\, \cos^2\varphi}\,, \tag{8}$$

where $\mu/M = \alpha/(1+\alpha)^2$ is the ratio of the reduced mass μ and the total mass M.

Lastly, we take the ratio of the momentum components (2) and (3) in order to eliminate the unknown v_1 and find

$$\tan\theta \;=\; \frac{v_2\, \sin\varphi}{\alpha\, u \;-\; v_2\, \cos\varphi}.$$

If we substitute Eq. (7), we easily obtain

$$\tan\theta \;=\; \frac{2\, \sin\varphi\, \cos\varphi}{1 \;+\; \alpha \;-\; 2\, \cos^2\varphi},$$

or

$$\theta(\varphi; \alpha) \;=\; \arctan\left(\frac{\sin 2\varphi}{\alpha \;-\; \cos 2\varphi} \right). \tag{9}$$

Two interesting limits of Eq. (9) are worth discussing. In the limit $\alpha \to 0$ (i.e., an infinitely massive target), we find

$$\varphi \;=\; \frac{1}{2}\, (\pi \;-\; \theta),$$

while in the limit $\alpha = 1$ (i.e., a collision involving identical particles), we find

$$\varphi \;=\; \frac{\pi}{2} \;-\; \theta.$$

Thus, the angular sum $\theta + \varphi$ for a like-particle collision is always 90 degrees.

We summarize by stating that, after the collision, the momenta $\mathbf{q}_1$ and $\mathbf{q}_2$ in the LAB frame (where m_2 is initially at rest) are

$$\mathbf{q}_1 = p \left[1 - \frac{4\alpha}{(1+\alpha)^2} \cos^2\varphi \right]^{1/2} (\cos\theta\,\widehat{\mathbf{x}} + \sin\theta\,\widehat{\mathbf{y}})$$

$$\mathbf{q}_2 = \frac{2p\cos\varphi}{1+\alpha} (\cos\varphi\,\widehat{\mathbf{x}} - \sin\varphi\,\widehat{\mathbf{y}})$$

where $\mathbf{p}_1 = p\,\widehat{\mathbf{x}}$ is the initial momentum of particle 1.

2 Two-Particle Collisions in the CM Frame

In the CM frame, the collision between particles 1 and 2 is described quite simply. Before the collision, the momenta of particles 1 and 2 are equal in magnitude but with opposite directions

$$\mathbf{p}'_1 = \mu u\,\widehat{\mathbf{x}} = -\,\mathbf{p}'_2,$$

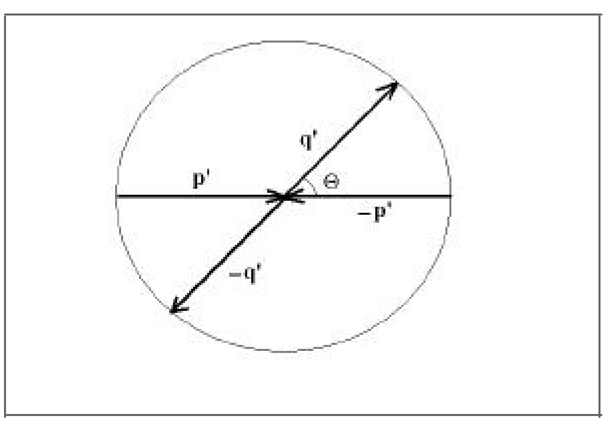
where μ is the reduced mass and a prime now denotes kinematic quantities in the CM frame.

After the collision, conservation of energy-momentum dictates that

$$\mathbf{q}'_1 = \mu u\,(\cos\Theta\widehat{\mathbf{x}} + \sin\Theta\,\widehat{\mathbf{y}}) = -\,\mathbf{q}'_2,$$

where Θ is the scattering angle in the CM frame and $\mu u = p/(1+\alpha)$. Thus the particle velocities after the collision in the CM frame are

$$\mathbf{v}'_1 = \frac{\mathbf{q}'_1}{m_1} = \frac{u}{1+\alpha} (\cos\Theta\widehat{\mathbf{x}} + \sin\Theta\,\widehat{\mathbf{y}}) \quad \text{and} \quad \mathbf{v}'_2 = -\,\alpha\,\mathbf{v}'_1.$$

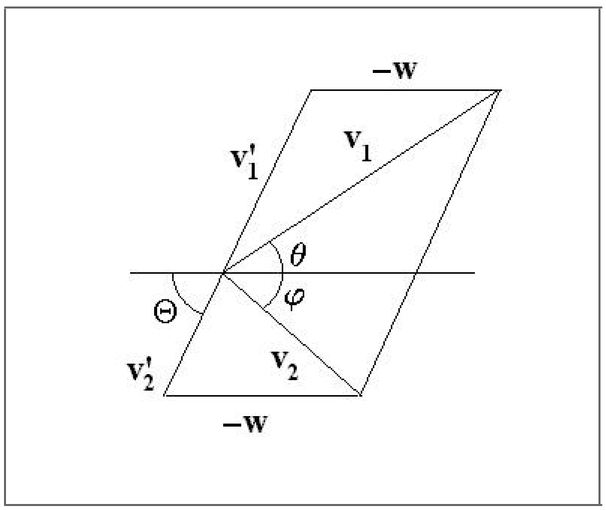


3 Connection between the CM Frame and the LAB Frame

We now establish the connection between the momenta $\mathbf{q}_1$ and $\mathbf{q}_2$ in the LAB frame and the momenta $\mathbf{q}_1'$ and $\mathbf{q}_2'$ in the CM frame. First, we denote the velocity of the CM as

$$\mathbf{w} \;=\; \frac{m_1\,\mathbf{u}_1 + m_2\,\mathbf{u}_2}{m_1 + m_2} \;=\; \frac{\alpha\,u}{1+\alpha}\,\widehat{\mathbf{x}},$$

so that $w = |\mathbf{w}| = \alpha\,u/(1+\alpha)$ and $|\mathbf{v}_2'| = w = \alpha\,|\mathbf{v}_1'|$.



The connection between $\mathbf{v}_1'$ and $\mathbf{v}_1$ is expressed as

$$\mathbf{v}_1' \;=\; \mathbf{v}_1 \;-\; \mathbf{w} \quad\rightarrow\quad \left\{ \begin{array}{rcl} v_1\,\cos\theta & = & w\,(1+\alpha^{-1}\,\cos\Theta) \\ \\ v_1\,\sin\theta & = & w\,\alpha^{-1}\,\sin\Theta \end{array} \right.$$

so that

$$\tan\theta \;=\; \frac{\sin\Theta}{\alpha + \cos\Theta},$$

and

$$v_1 \;=\; v_1'\,\sqrt{1 \;+\; \alpha^2 \;+\; 2\,\alpha\,\cos\Theta},$$

where $v_1' = u/(1+\alpha)$.

Likewise, the connection between $\mathbf{v}_2'$ and $\mathbf{v}_2$ is expressed as

$$\mathbf{v}_2' \;=\; \mathbf{v}_2 \;-\; \mathbf{w} \quad\rightarrow\quad \left\{ \begin{array}{rcl} v_2\,\cos\varphi & = & w\,(1-\cos\Theta) \\ \\ v_2\,\sin\varphi & = & w\,\sin\Theta \end{array} \right.$$

so that

$$\tan\varphi \;=\; \frac{\sin\Theta}{1-\cos\Theta} \;=\; \cot\frac{\Theta}{2} \quad\rightarrow\quad \varphi \;=\; \frac{1}{2}\,(\pi \;-\; \Theta),$$

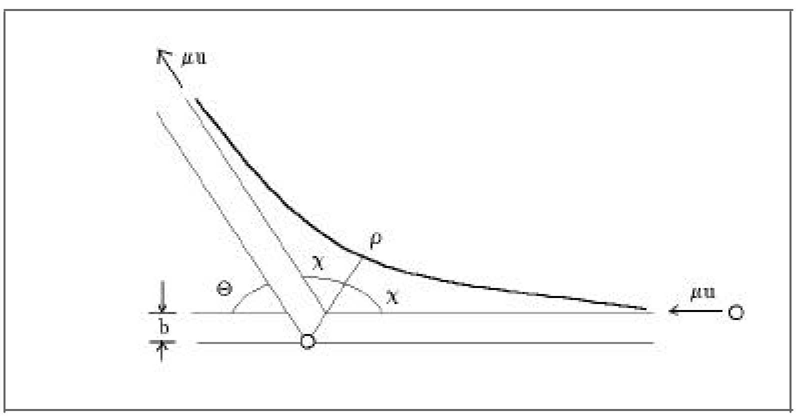
and

$$v_2 \;=\; 2\,v_2'\;\sin\frac{\Theta}{2},$$

where $v_2' = \alpha\,u/(1+\alpha)$.

4 Scattering Cross Sections

We consider the case of a projectile particle of mass μ being deflected by a repulsive central-force potential $U(r) > 0$. As the projectile particle approaches from the right (at $r = \infty$ and $\theta = 0$) moving with speed u, it is progressively deflected until it reaches a minimum radius ρ at $\theta = \chi$ after which the projectile particle moves away from the repulsion center until it reaches $r = \infty$ at a deflection angle $\theta = \Theta$ and again moving with speed u. From the Figure shown below, we can see that the scattering process is symmetric about the line of closest approach.



The angle of closest approach

$$\chi \;=\; \frac{1}{2}\,(\pi - \Theta) \tag{10}$$

is a function of the distance of closest approach ρ, the total energy E, and the angular momentum ℓ. The distance ρ is, of course, a turning point ($\dot{r} = 0$) and is the only root of the equation

$$E' \;=\; U(\rho) \;+\; \frac{\ell^2}{2\mu\,\rho^2}, \tag{11}$$

where $E' = \mu\,u^2/2$ is the total initial energy of the projectile particle in the CM frame.

The path of the projectile particle in the Figure above is labeled by the impact parameter b (the distance of closest approach in the case of no interaction, $U = 0$) and a simple calculation shows that the angular momentum is

$$\ell \;=\; \mu u\, b \;=\; \sqrt{2\mu\, E'}\, b. \tag{12}$$

It is thus quite clear that ρ is a function of E', μ, and b. Hence, the angle χ is defined in terms of the standard integral

$$\chi \;=\; \int_\rho^\infty \frac{(\ell/r^2)\, dr}{\sqrt{2\mu\,[E' - U(r)] \;-\; (\ell^2/r^2)}} \;=\; \int_\rho^\infty \frac{(b/r^2)\, dr}{\sqrt{1 \;-\; (b^2/r^2) \;-\; U(r)/E'}}. \tag{13}$$

Once an expression $\Theta(b, E')$ is obtained from Eq. (13), we may invert it to obtain $b(\Theta, E')$.

5 Scattering Cross Sections in CM and LAB Frames

The infinitesimal cross section $d\sigma'$ in the CM frame is defined in terms of $b(\Theta, E')$ as

$$d\sigma'(\Theta, E') \;=\; \pi\; db^2(\Theta, E').$$

Using the relation (10), the impact parameter $b(\Theta)$ is thus a function of the deflection angle Θ (at fixed energy) and the differential cross section in the CM frame is defined as

$$\sigma'(\Theta) \;=\; \frac{d\sigma'}{2\pi\, d(\cos\Theta)} \;=\; \frac{b(\Theta)}{\sin\Theta} \left| \frac{db(\Theta)}{d\Theta} \right|. \tag{14}$$

The differential cross section can also be written in the LAB frame in terms of the angle θ as

$$\sigma(\theta) \;=\; \frac{d\sigma}{2\pi\, d(\cos\theta)} \;=\; \frac{b(\theta)}{\sin\theta} \left| \frac{db(\theta)}{d\theta} \right|. \tag{15}$$

Since the infinitesimal cross section $d\sigma = d\sigma'$ is the same in both frames, we find

$$\sigma(\theta)\; \sin\theta\, d\theta \;=\; \sigma'(\Theta)\; \sin\Theta\, d\Theta,$$

from which we obtain

$$\sigma(\theta) \;=\; \sigma'(\Theta)\, \frac{\sin\Theta}{\sin\theta}\, \frac{d\Theta}{d\theta}, \tag{16}$$

or

$$\sigma'(\Theta) \;=\; \sigma(\theta)\, \frac{\sin\theta}{\sin\Theta}\, \frac{d\theta}{d\Theta}. \tag{17}$$

Eq. (16) yields an expression for the differential cross section in the LAB frame $\sigma(\theta)$ once the differential cross section in the CM frame $\sigma'(\Theta)$ and an explicit formula for $\Theta(\theta)$ are known. Eq. (17) represents the inverse transformation $\sigma(\theta) \to \sigma'(\Theta)$.

These transformations rely on finding relations between the LAB deflection angle θ and the CM deflection angle Θ. We found earlier the relation

$$\tan\theta \;=\; \frac{\sin\Theta}{\alpha + \cos\Theta}, \tag{18}$$

where $\alpha = m_1/m_2$, which can be converted into

$$\sin(\Theta - \theta) \;=\; \alpha\,\sin\theta. \tag{19}$$

We now show how to obtain an expression for $(\sin\Theta/\sin\theta)\,d\Theta/d\theta$ by using Eqs. (18) and (19). First, we use Eq. (19) to obtain

$$\frac{d\Theta}{d\theta} \;=\; \frac{\alpha\,\cos\theta \;+\; \cos(\Theta-\theta)}{\cos(\Theta-\theta)}, \tag{20}$$

where

$$\cos(\Theta-\theta) \;=\; \sqrt{1 \;-\; \alpha^2\,\sin^2\theta}\;.$$

Next, using Eq. (18), we show that

$$\begin{aligned}\frac{\sin\Theta}{\sin\theta} &= \frac{\alpha+\cos\Theta}{\cos\theta} = \frac{\alpha \;+\; [\cos(\Theta-\theta)\,\cos\theta \;-\; \overbrace{\sin(\Theta-\theta)}^{=\,\alpha\,\sin\theta}\,\sin\theta]}{\cos\theta} \\ &= \frac{\alpha\,(1-\sin^2\theta) \;+\; \cos(\Theta-\theta)\,\cos\theta}{\cos\theta} = \alpha\,\cos\theta \;+\; \sqrt{1 \;-\; \alpha^2\,\sin^2\theta}\;. \end{aligned}\tag{21}$$

Thus by combining Eqs. (20) and (21), we find

$$\frac{\sin\Theta}{\sin\theta}\,\frac{d\Theta}{d\theta} \;=\; \frac{[\alpha\,\cos\theta \;+\; \sqrt{1 \;-\; \alpha^2\,\sin^2\theta}]^2}{\sqrt{1 \;-\; \alpha^2\,\sin^2\theta}} \;=\; 2\,\alpha\,\cos\theta \;+\; \frac{1+\alpha^2\,\cos 2\theta}{\sqrt{1 \;-\; \alpha^2\,\sin^2\theta}}, \tag{22}$$

which is valid for $\alpha < 1$. Lastly, noting from Eq. (19)

$$\Theta(\theta) \;=\; \theta \;+\; \arcsin(\,\alpha\,\sin\theta\,),$$

the transformation $\sigma'(\Theta) \rightarrow \sigma(\theta)$ is now complete. Similar manipulations yield the transformation $\sigma(\theta) \rightarrow \sigma'(\Theta)$. We note that the LAB-frame cross section $\sigma(\theta)$ are generally difficult to obtain for arbitrary mass ratio $\alpha = m_1/m_2$.

6 Rutherford Scattering

As an explicit example of the formalism of scattering cross section, we now investigate the scattering of a charged particle of mass m_1 and charge q_1 by another charged particle of mass m_2 and charge q_2 such that $q_1 \cdot q_2 > 0$. This situation leads to the two particles experiencing a repulsive central force with potential

$$U(r) \;=\; \frac{k}{r},$$

where $k = q_1 q_2/4\pi\,\varepsilon_0 > 0$. The single turning point in this case is the distance of closest approach

$$\rho \;=\; r_0 \;+\; \sqrt{r_0^2 + b^2} \;=\; r_0\,(1+\epsilon), \tag{23}$$

where $2\,r_0 = k/E'$ is the distance of closest approach for a head-on collision (for which the impact parameter b is zero). The problem of the electrostatic repulsive interaction between a positively-charged alpha particle (i.e., the nucleus of a Helium atom) and positively-charged nucleus of a gold atom was first studied by Rutherford and the scattering cross section for this problem is known as the Rutherford cross section.

The angle χ at which the distance of closest approach is reached is calculated from Eq. (13) as

$$\chi \;=\; \int_\rho^\infty \frac{(b/r)\;dr}{\sqrt{r^2 - 2\,r_0 r - b^2}},$$

which can be converted through the substitution $r = r_0\,(1 + \epsilon\;\sec\psi)$ to the integral

$$\chi(\epsilon) \;=\; \sqrt{\epsilon^2 - 1}\;\int_0^{\pi/2} \frac{d\psi}{\epsilon + \cos\psi}, \tag{24}$$

where

$$\frac{b}{r_0} \;=\; \sqrt{\frac{2\,E'\ell^2}{\mu\,k^2}} \;=\; \sqrt{\epsilon^2 - 1}.$$

By using the integral formula

$$\int \frac{d\psi}{\epsilon + \cos\psi} \;=\; \frac{2}{\sqrt{\epsilon^2 - 1}}\;\arctan\left(\sqrt{\frac{\epsilon - 1}{\epsilon + 1}}\;\tan(\psi/2)\right),$$

we easily find

$$\chi \;=\; 2\;\arctan\left(\sqrt{\frac{\epsilon - 1}{\epsilon + 1}}\right),$$

or

$$\epsilon \;=\; \frac{1 + \tan^2(\chi/2)}{1 - \tan^2(\chi/2)} \;=\; \frac{1}{\cos^2(\chi/2) - \sin^2(\chi/2)} \;=\; \sec\chi.$$

Since $b/r_0 = \sqrt{\epsilon^2 - 1}$, we now find

$$\frac{b}{r_0} \;=\; \tan\chi. \tag{25}$$

Using the relation (10), we find

$$b(\Theta) \;=\; r_0\;\cot\frac{\Theta}{2}, \tag{26}$$

and thus $db(\Theta)/d\Theta = -\,(r_0/2)\;\csc^2(\Theta/2)$. The CM cross section is

$$\sigma'(\Theta) \;=\; \frac{b(\Theta)}{\sin\Theta}\left|\frac{db(\Theta)}{d\Theta}\right| \;=\; \frac{r_0^2}{4\;\sin^4(\Theta/2)},$$

or

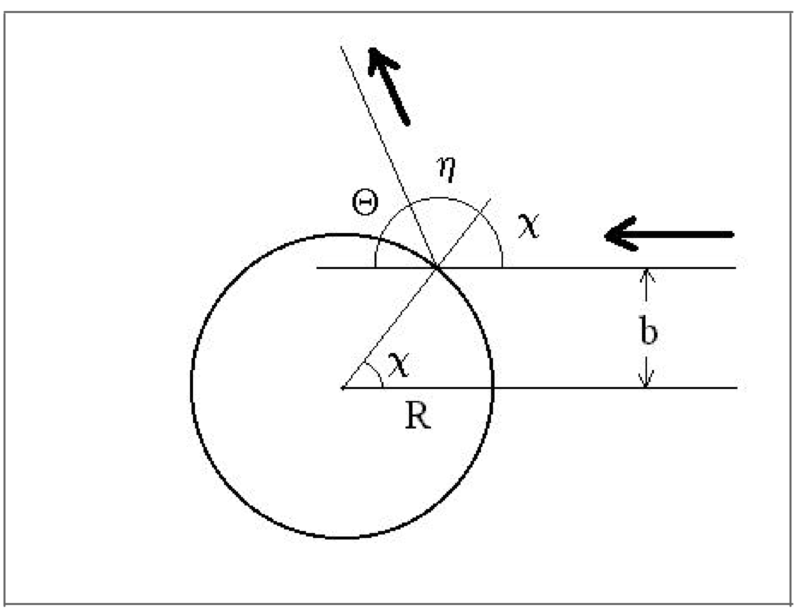
$$\sigma'(\Theta) \;=\; \left(\frac{k}{4E'\;\sin^2(\Theta/2)}\right)^2. \tag{27}$$

Note that the Rutherford scattering cross section does not depend on the sign of k and is thus valid for both repulsive and attractive interactions. Moreover, we note that the Rutherford scattering cross section becomes very large in the forward direction $\Theta \to 0$ (where $\sigma' \to \Theta^{-4}$) while the differential cross section as $\Theta \to \pi$ behaves as $\sigma' \to (k/4E')^2$.

7 Hard-Sphere and Soft-Sphere Scattering

7.1 Hard-Sphere Scattering

Let us consider the collision of a point-like particle of mass m_1 with a hard sphere of mass m_2 and radius R.



From the Figure above, we see that the impact parameter is

$$b \;=\; R\,\sin\chi, \tag{28}$$

where χ is the angle of incidence. The angle of reflection η is different from the angle of incidence χ for the case of arbitrary mass ratio $\alpha = m_1/m_2$. To show this, we decompose the velocities in terms of components perpendicular and tangential to the surface of the sphere at the point of impact, i.e., we respectively find

$$\begin{aligned}\alpha\,u\,\cos\chi &= v_2 \;-\; \alpha\,v_1\,\cos\eta \\ \alpha\,u\,\sin\chi &= \alpha\,v_1\,\sin\eta.\end{aligned}$$

From these expressions we obtain

$$\tan\eta \;=\; \frac{\alpha\,u\,\sin\chi}{v_2 - \alpha\,u\,\cos\chi}.$$

From the Figure above, we find the deflection angle $\theta = \pi - (\chi + \eta)$ and the recoil angle $\varphi = \chi$ and thus,

$$v_2 \;=\; \left(\frac{2\,\alpha}{1+\alpha}\right) u\,\cos\chi,$$

and thus

$$\tan\eta \;=\; \left(\frac{1+\alpha}{1-\alpha}\right)\tan\chi. \tag{29}$$

We, therefore, easily see that $\eta = \chi$ (the standard form of the Law of Reflection) only if $\alpha = 0$ (i.e., the target particle is infinitely massive).

In the CM frame, the collision is symmetric with a deflection angle $\chi = \frac{1}{2}(\pi - \Theta)$, so that

$$b \;=\; R\sin\chi \;=\; R\,\cos\frac{\Theta}{2}.$$

The scattering cross section in the CM frame is

$$\sigma(\Theta) \;=\; \frac{b(\Theta)}{\sin\Theta}\left|\frac{db(\Theta)}{d\Theta}\right| \;=\; \frac{R\,\cos(\Theta/2)}{\sin\Theta}\cdot\left|-\frac{R}{2}\,\sin(\Theta/2)\right| \;=\; \frac{R^2}{4}, \tag{30}$$

and the total cross section is

$$\sigma_T \;=\; 2\pi\int_0^{\pi}\sigma(\Theta)\,\sin\Theta\,d\Theta \;=\; \pi\,R^2,$$

i.e., the total cross section for the problem of hard-sphere collision is equal to the effective area of the sphere.

The scattering cross section in the LAB frame can also be obtained for the case $\alpha < 1$ using Eqs. (16) and (22) as

$$\sigma(\theta) \;=\; \frac{R^2}{4}\left(2\,\alpha\,\cos\theta \;+\; \frac{1+\alpha^2\cos 2\theta}{\sqrt{1\;-\;\alpha^2\sin^2\theta}}\right), \tag{31}$$

for $\alpha = m_1/m_2 < 1$. The integration of this formula must yield the total cross section

$$\sigma_T \;=\; 2\pi \int_0^{\pi} \sigma(\theta)\,\sin\theta\, d\theta,$$

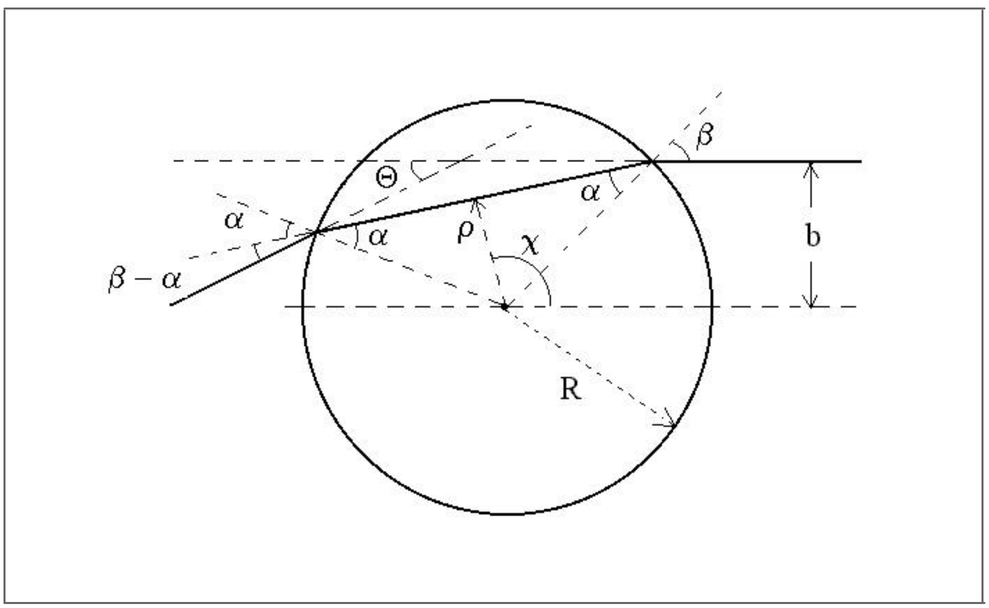
where $\theta_{max} = \pi$ for $\alpha < 1$.

7.2 Soft-Sphere Scattering

Let us now consider the scattering of a particle subjected to the following attractive potential

$$U(r) \;=\; \begin{cases} -U_0 & \text{for} \quad r \;<\; R \\ \\ 0 & \text{for} \quad r \;>\; R \end{cases} \tag{32}$$

where the constant U_0 denotes the depth of the attractive potential well. We denote β the angle at which the incoming particle enters the **soft-sphere** potential (see Figure below), and thus the impact parameter b of the incoming particle is $b = R\,\sin\beta$.



The particle enters the soft-sphere potential region ($r < R$) and reaches a distance of closest approach ρ, defined from the turning-point condition

$$E \;=\; -\,U_0 \;+\; E\,\frac{b^2}{\rho^2} \quad\rightarrow\quad \rho \;=\; \frac{b}{\sqrt{1 + U_0/E}} \;=\; \frac{R}{n}\,\sin\beta,$$

where $n = \sqrt{1 + U_0/E}$ denotes the **index of refraction** of the soft-sphere potential region. From the Figure above, we note that an optical analogy helps us determine that, through

Snell's law, we find

$$\sin\beta \;=\; n\,\sin\left(\beta \;-\; \frac{\Theta}{2}\right), \tag{33}$$

where the **transmission** angle α is given in terms of the **incident** angle β and the CM scattering angle $-\Theta$ as $\Theta = 2\,(\beta - \alpha)$.

The distance of closest approach is reached at an angle χ is determined as

$$\begin{aligned} \chi &= \beta \;+\; \int_{\rho}^{R} \frac{b\,dr}{r\,\sqrt{n^2 r^2 - b^2}} \\ &= \beta \;+\; \arccos\left(\frac{b}{nR}\right) \;-\; \underbrace{\arccos\left(\frac{b}{n\rho}\right)}_{=\,0} \\ &= \beta \;+\; \arccos\left(\frac{b}{nR}\right) \;=\; \frac{1}{2}\,(\pi + \Theta), \end{aligned} \tag{34}$$

and, thus, the impact parameter $b(\Theta)$ can be expressed as

$$b(\Theta) \;=\; nR\,\sin\left(\beta(b) \;-\; \frac{\Theta}{2}\right) \quad\rightarrow\quad b(\Theta) \;=\; \frac{nR\,\sin(\Theta/2)}{\sqrt{1 + n^2 - 2n\,\cos(\Theta/2)}}. \tag{35}$$

The opposite case of a repulsive soft-sphere potential, where $-U_0$ is replaced with U_0 in Eq. (32), is treated by replacing $n = (1 + U_0/E)^{\frac{1}{2}}$ with $n = (1 - U_0/E)^{-\frac{1}{2}}$ and Eq. (35) is replaced with

$$b(\Theta) \;=\; n^{-1}\,R\,\sin\left(\beta(b) \;+\; \frac{\Theta}{2}\right) \quad\rightarrow\quad b(\Theta) \;=\; \frac{R\,\sin(\Theta/2)}{\sqrt{1 + n^2 - 2n\,\cos(\Theta/2)}}, \tag{36}$$

while Snell's law (33) is replaced with

$$\sin\left(\beta \;+\; \frac{\Theta}{2}\right) \;=\; n\,\sin\beta.$$